Estimating the U.S. Treasury Term Structure of Interest Rates

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The term structure of interest rates is the set of interest rates applying over all possible maturities. The current term structure can define the state of the bond market, identify rich and cheap instruments, price nontraded issues, and even record the market’s overall assessment of future interest rate movements. A term structure history can quantify the interest rate risk present in the bond market. The term structure of interest rates plays the central role in bond market analysis.

In theory, the set of bond prices at any instant in time, uniquely determines the term structure. In practice, this is not true. Not all bond prices lie on the term structure, and the market does not span all possible maturities. Different estimation procedures and setups will fit different term structures to the same set of market prices. The art of term structure estimation involves clearly identifying specific objectives for the term structure, and choosing an estimation procedure consistent with those objectives.

This chapter will discuss the consideration and alternatives for estimating U.S. Treasury term structures. The second section will compare the concept of the Treasury term structure to the traditional Treasury yield curve. The third section will discuss the important considerations behind choosing a term structure estimation procedure and the fourth section will present estimation examples.

THE TERM STRUCTURE IS NOT THE YIELD CURVE

The Treasury yield curve is the traditional measure of interest rates in the Treasury market. Every Treasury issue has
its own particular yield. The semiannual yield to maturity is the unique solution to:

\[ P_i(t) = \sum_{t} \frac{c_i(T)}{1 + \left(\frac{y_i(t)}{2}\right)^{2(T-t)}} \]  

(1)

where

- \( t \) = current date
- \( P_i(t) \) = current market price for Treasury issue \( i \).
- \( c_i(T) \) = cash flow for Treasury issue \( i \) at time \( T \)
- \( y_i(t) \) = current semiannual yield to maturity for Treasury issue \( i \).

This definition implicitly assumes a constant yield applying over the entire maturity of each issue. The Treasury field curve is then a scatter plot of these bond specific yields versus the bond specific maturity. Exhibit 1 illustrates the U.S. Treasury yield curve for July 29, 1988.

EXHIBIT 1
TREASURY YIELD CURVE: JULY 29, 1988

The key insight of term structure analysis is to separate bond specific characteristics from marketwide properties. While yield is a bond specific property, the term structure of
interest rates defines a marketwide property: the set of interest rates operating at a given instant in time. The bond specific properties are the specific cash flows of the issue. Hence:

\[ P_i(t) = \sum_T \frac{c_f(T)}{s(t, T)} \gamma^{2(T-i)} + \varepsilon_i \]  

(2)

where

\[ s(t, T) = \] semiannual spot rate between \( t \) and \( T \)

\[ \varepsilon_i = \] pricing error for Treasury issue \( i \)

The pricing error \( \varepsilon_i \) arises in Equation 2, because the spot rate \( s(t, T) \) is the same for all Treasury issues, and, in general, cannot fit every price exactly. Compare this to the situation described by Equation 1, where each Treasury issue has its own yield, defined so that the pricing error is zero.

These pricing errors are a consequence of the marketwide term structure approach, and can be quite useful. The term structure is a uniform market property, but the Treasury market is not perfectly homogeneous. The estimated term structure will identify rich and cheap Treasury instruments (those with positive or negative \( \varepsilon_i \), respectively). These mispricings arise either from heterogeneities or market inefficiencies.

Equation 2 describes the term structure of semiannual Treasury spot rates \( s(t, T) \). There exists other equivalent ways to describe the term structure, including continuous spot rates, pure discount bond prices, and forward rates. The continuous spot rates \( s_c(t, T) \) are defined by:

\[ P_i(t) = \sum_T c_f(T) \cdot e^{-\varepsilon_i(t, T)(T-i)} + \varepsilon_i. \]  

(3)

Each spot rate uniquely corresponds to a pure discount bond price:

\[ PDB(t, T) = e^{-\varepsilon_i(t, T)(T-i)}. \]  

(4)

Equation 4 describes the price, at time \( t \), of an instrument paying $1.00 at time \( T \). Each continuous spot rate \( s_c(t, T) \) applies between the same initial time \( t \) and different final times \( T \). A set of continuous forward rates \( f_c(t, T) \) equivalently describes rates, but between future time \( T_1 \) and \( T_2 \):
\[ PDB(t, T_2) = PDB(t, T_1) \cdot e^{-f_c(t, T_1, T_2)(T_2 - T_1)} \]  \hspace{1cm} (5)

or:

\[ S_c(t, T_2)(T_2 - t) = S_c(t, T_1)(T_1 - t) + f_c(t, T_1, T_2)(T_2 - T_1) \]  \hspace{1cm} (6)

and:

\[ S_c(t, T)(T - t) = \sum_{t_{sr}} f_c(t, T_{i-1}, T_i)(T_i - T_{i-1}) \]  \hspace{1cm} (7)

While the spot rates are more intuitive, because they are analogous to yields, and pure discount bond prices are linearly related to observed bond prices, the term structure estimation techniques described here focus on the forward rates. The forward rates are the independent set of rates: as Equation 7 demonstrates, spot rates are averages of forward rates. Since forward rates, spot rates, and pure discount bond prices are all equivalent term structure descriptions, the spot rates and pure discount bond prices will immediately follow from the set of estimated forward rates.

**TERM STRUCTURE ESTIMATION CONSIDERATIONS**

The term structure will not price all bonds exactly. The discrete set of market prices and bond cash flows do not uniquely determine a continuous term structure. Different estimation procedures, all attempting to minimize overall pricing error, will lead to different term structures. Choosing a particular technique will require matching that procedure to the specific analysis goal. Three general considerations of importance for this choice are choice of universe, closeness of fit, and robustness.

Given the objective of estimating a Treasury term structure, the choice of universe for the estimation procedure still remains. Since there exist heterogeneities within the full Treasury universe, many objectives warrant restricting this universe for the estimation procedure. For example, flower bonds differ from all other Treasury instruments in their particular estate tax consequences. Estimates of Treasury term structures often exclude the influence of flower bonds. If the objective of the term structure is a snapshot of the actively
traded Treasury market, then the estimation universe should, perhaps, consist only of the most recent issues.

Closeness of fit is an essential consideration for term structure estimation. While the term structure is, in principle, continuous, the term structure estimation will focus on estimating a discrete set of forward rates between particular vertices, with interpolated values between. For example, one could choose the five vertices \( \{0, 1, 5, 10, 30\} \), and then estimate four constant forward rates between 0 and 1 year, 1 and 5 years, 5 and 10 years, and 10 and 30 years. The term structure would consist of those four forward rates. Of course, it will probably prove impossible to closely fit all 200 or so Treasury prices by adjusting only four forward rates. The more vertices, the more degrees of freedom with which to fit the observed bond prices. Consequently, the question of closeness of fit is a question of number of vertices.

At the extremes, one can choose just two vertices: 0 and 30 years, leading to a flat term structure; or a number of vertices equal to the number of bonds, coming close to pricing all bonds exactly. In either case, the resulting term structure contains little new information. A flat term structure is too coarse to realistically represent the Treasury market. A 200 vertex term structure will contain no more information than the 200 or so bond prices. Such a finely defined term structure will not even fairly price nontraded issues, because it is likely to be quite irregular and subject to estimation errors, as discussed below.

For the purpose of identifying relative value within the Treasury market, the number of term structure vertices should match the natural sectors in the market. If market participants consider all Treasury instruments with 15-20 remaining years of maturity interchangeable, then the natural vertices occur at 15 to 20 years, with none between. Price discrepancies may then identify opportunities. A flat term structure will identify rich and cheap issues, but if investors do not consider T-bills and 30 year bonds interchangeable, then a relative mispricing does not identify an opportunity.

For the purpose of pricing nontraded issues, the term structure should closely fit bond prices. The number of vertices should, therefore, be large, but it still should not exceed the
natural number of maturity sectors in the market, in order to
accurately estimate the market’s view of the value of a
nontraded issue. Also, though mean pricing errors always
decrease as the number of vertices increases, the resulting term
structure is more vulnerable to data errors, because fewer bond
prices will influence each forward rate.

Considerations of closeness of fit affect the handling of
anomalies. Should anomalies appear in the term structure
itself, or as price errors? Term structure anomalies, by
definition, do not imply arbitrage opportunities. Price errors do
imply opportunities. A natural Treasury market vertex occurs
in every maturity sector of interchangeable instruments. This
choice of vertices should insure that arbitrage opportunities
appear as price errors, while other structural anomalies appear
in the term structure itself.

For the purpose of estimating interest rate risk, the required
number of vertices may be quite small. Most term structure
risk consists of just three types of term structure movements:
shift, twist, and butterfly. Four adroitly chosen vertices, with
their three associated forward rates could capture all three
movements, because changes in these three rates will
decompose into a shift, a twist, and a butterfly. However,
estimating risk arising from changing anomalies requires more
vertices.

Robustness is the final important consideration for
choosing an estimation procedure. In contrast to choice of
universe and closeness of fit, robustness is a numerical rather
than a financial consideration. Regardless of the financial
analysis objective, more robustness is, ceteris paribus, always
better. Robustness is simply the insensitivity of the algorithm
to individual pricing errors in the data. Robustness is of crucial
practical importance, because of bad data. These can generate
large pricing anomalies. A robust estimation will minimize
their influence. Generally speaking, as the number of vertices
increases, the robustness of the estimation will decrease,
because fewer bond prices will influence each forward rate.
TERM STRUCTURE ESTIMATION EXAMPLES

This section will present three term structure estimation examples, with particular emphasis on the questions of closeness of fit and robustness. The universe of non-flower Treasury instruments on July 29, 1988 will constitute the estimation universe for all these examples. These three examples all begin with the same set of bond prices. The term structure goals differ, however, and the three different problem setups and estimation techniques lead to three different term structures.

The goal of Example 1 is to estimate a Treasury term structure which closely fits bond prices while still identifying relative value. Hence, this example involves a 14 vertex term structure.

Example 1 Vertices
0.00
0.25
0.50
1.00
2.00
3.00
4.00
5.00
7.00
10.00
15.00
20.00
25.00
30.00

The estimated forward rates in this example arise through a standard regression technique. Regression solves for the set of continuous forward rates, which minimizes the total squared pricing errors:
\[
\hat{P}_t(t) = \sum_T c_f(T) \cdot PDB(t, T) \\
= \sum_T c_f(T) \cdot \exp\left\{ -\sum_{T_j} f_c(T_j, T_j)(T_j - T_j)\right\} 
\]  
(8)

\[
= \text{fitted price.}
\]

Total Squared Pricing Error = \[\sum_i \left( \hat{P}_i(t) - P_i(t) \right)^2 \] (9)

Regression solves for the set of \( f_c(t, T_{j-1}, T_j) \), which minimizes the right hand side of Equation 9.

In general, this regression technique is quite flexible. One can choose a variety of weighting schemes, or minimize relative instead of dollar pricing error. Extensive software and statistical support accompany this algorithm. Unfortunately, its definition, in terms of squared errors, makes it quite sensitive to anomalies.

Exhibit 2 illustrates the Example 1 term structure. The RMS relative pricing error\(^2\) is only 22 basis points, though the long forward rates illustrate an anomaly. The anomalous dip at the long end of the term structure may well arise due to a short supply of 30-year bonds. Because regression is sensitive to outliers, and because the term structure contains many long vertices, this supply anomaly appears directly in the term structure itself. The 20 to 25 year forward rate and the 25 to 30 year forward rate are independent in this example, so the term structure itself absorbs the mispricing between the 20 to 25 year maturity sector and the 25 to 30 year maturity sector.

The goal of Example 2 is to estimate a parsimonious term structure, with less importance placed on closeness of fit. This example involves only 7 vertices:
Example 2 Vertices

0.00
1.00
3.00
5.00
7.00
15.00
30.00

This parsimonious term structure should adequately define the Treasury term structure for risk modeling purposes. Once again, the estimated forward rates arise through regression.

Exhibit 3 plots the Example 2 term structure. The RMS relative pricing error is now 34 basis points. Here the long maturity anomaly has disappeared. This term structure will identify the 30-year bond as rich—the anomaly appears in the pricing error and not in the term structure itself.

EXHIBIT 2
TREASURY TERM STRUCTURE: JULY 29, 1988

Source: BARRA
Examples 1 and 2 use standard regression to estimate the term structure. But regression suffers from an exaggerated sensitivity to outliers. It is not very robust. The goal of Example 3 is to more robustly estimate a term structure using the Example 1 vertices.

The estimation method in Example 3 will use arbitrage-free forward rate priors to increase the robustness of the standard regression approach. In essence, the idea is simple: term structure estimation should rely not just on bond prices, but on all possible information concerning the term structure. Theoretical models of the term structure imply that forward rates must obey certain relationships to preclude arbitrage opportunities between bonds of different maturities. These theoretical relationships can influence term structure estimation, though the resulting term structure will not exactly obey the relationships, just like it will not exactly price all bonds.

The forward rate priors impose these relationships between the various forward rates. These relationships can arise from a particular arbitrage-free term structure model, or they can represent an empirical generalization of these theoretical
models. For example, the priors may simply state that adjacent forward rates should be equal.

These considerations—prior assumptions about the term structure—add stability to the estimation. The new estimation procedure minimizes both pricing errors and forward prior errors:

\[
\text{Total General Error} = \sum_i \left( \hat{P}_i - P_i \right)^2 + \alpha \sum_j \left[ \hat{f}_c(t, T_{j-1}, T_j) - f_c(t, T_{j-1}, T_j) \right]^2
\]  

(10)

where \( \hat{f}_c(t, T_{j-1}, T_j) \) is the prior on the forward rate \( f_c(t, T_{j-1}, T_j) \), and depends upon the other forward rates \( f_c(t, T_{k-1}, T_k) \). The constant \( \alpha \) controls the relative weighting of the forward prior fit versus the bond price fit.

Exhibit 4 illustrates the Example 3 term structure. The RMS relative pricing error is 31 basis points. The 25 to 30 year forward rate shows some sign of the supply effect at 30 years, but the anomaly is much less pronounced than in Example 1. At the same time, the Example 3 term structure exhibits more structure than the parsimonious Example 2 term structure.
Including priors in the regression has two important benefits. First, the resulting estimation is more robust. The estimated forward rates depend on both price errors and forward prior errors, reducing their sensitivity to anomalous or incorrect prices. Second, imposing prior conditions on the forward rates can help identify arbitrage opportunities. If the observed term structure moves toward that theoretical shape, then this estimation technique will have accurately identified arbitrage opportunities. This second benefit, of course, depends on suitable choices for the forward priors. Setting the parameter $\alpha$ in Equation 10 to a low value will minimize the dependence on the particular forward prior choices, while retaining much of the increased robustness of the technique.

SUMMARY

The term structure of interest rates plays the central role in U.S. Treasury market analysis. Term structure analysis surpasses yield curve analysis by clearly separating marketwide properties from bond specific characteristics. However, the U.S. Treasury term structure is not uniquely defined. Actual estimated term structures depend both on observed bond prices, and on the estimation setup and
technique. Important considerations include choice of universe, closeness of fit, and robustness.

The three example term structures presented in this chapter all arose from the same set of bond prices. These examples involved different ultimate goals, and, hence, different setups and techniques. There exists no “correct” single term structure. There only exists correctly estimated term structures for particular specified goals. The art of term structure estimation involves clearly identifying goals, and then matching those goals to the proper estimation technique.


2 RMS Relative Pricing Error

\[ \text{RMS Relative Pricing Error} = \sqrt{\frac{1}{N} \sum_i \left( \frac{\hat{P}_i(t)}{P_i(t)} - 1 \right)^2} \]

3 The particular technique described here was developed by BARRA.


5 More elaborate weighting schemes are possible. For example, the weighting parameter \( \alpha \) in Equation 10 could depend upon index \( j \).