FIXED INCOME RISK MODELING

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Many years ago, bonds were boring. Returns were small and steady. Fixed income risk monitoring consisted of watching duration and avoiding low qualities. But as interest-rate volatility has increased and the variety of fixed income instruments has grown, both opportunities and dangers have flourished. Accurate fixed income risk measurement has become both more important and more difficult. The sources of fixed income risk have proliferated and intensified. Exposures to these risks are subtle and complex to estimate. Today’s fixed income environment requires advanced multifactor techniques to adequately model the many sources of risk influencing the market and powerful tools to compute exposures to those risks.

Duration is the traditional fixed income risk factor and measures exposure to the risk of parallel term-structure movements. But term structures not only shift in parallel, they also twist and bend; and these movements tend to increase in magnitude as interest rates rise. In addition to interest-rate volatility, most issues are exposed to various sources of default risk, assessed by marketwide sector and quality spreads. These spreads can depend on maturity and move unpredictably over time. Beyond marketwide sources of default risk, individual issues face specific sources of default risk.

Nominal cash flows and quality ratings no longer suffice to measure risk exposures. Call and put options and sinking-fund provisions can significantly alter an instrument’s risk exposures in intricate ways. Mortgage-backed securities are subject to uncertain prepayments, which influence the risk exposures of those instruments. When packaged as IOs, POs, or CMOs, the risk exposure accounting becomes even more difficult.

There is no question that building a fixed income risk model is complicated business: Forecasting risk factor
covariance and analyzing the Byzantine provisions of today’s fixed income instruments require sophisticated methods.

Using a fixed income risk model, however, should be intuitive and straightforward. Bond investors should find the risk factors sensible. Risk analysis results should be precise, but still conform to investor instincts. A good risk model should actually simplify the investment process, quantify risks, and increase investor insight.

Fixed income risk modeling plays a critical role in bond portfolio management, underlying benchmark tracking, immunization, active strategy implementation, and performance measurement and analysis. Benchmark tracking involves comparing the risk exposures of an investment portfolio and a benchmark. Matching those exposures should lead to investment returns that accurately track benchmark returns. Immunization involves comparing the risk exposures of a portfolio and a liability stream. Matching those exposures should immunize the portfolio’s liability coverage against market changes. Active strategies involve deliberate risk exposures relative to a benchmark, aimed at exceeding benchmark returns. Performance measurement and analysis involves identifying active bets and studying their past performance so as to measure bond manager skill.

This chapter describes a multifactor approach to risk modeling. This approach consists of two basic components. First, a valuation model identifies and values the many risk factors in the market. The valuation model requires the machinery to estimate exposures to these risk factors, including an option simulation to handle the wide variety of optionable fixed income securities. Second, a risk model examines the historical behavior of these risk factors to estimate their variances and covariances. While the presentation here will be general, this chapter will conclude with evidence of the performance of multifactor risk models based on their specific application to the U.S. bond market.¹
THE VALUATION MODEL

The following multifactor valuation model is designed to identify and value risk factors in the market. This model estimates bond prices as

\[
PM_n(t) = \sum_{T} \frac{cf_n(T) \cdot PDB(t, T)}{\exp[\kappa_n(t) \cdot T]} + \xi_n(t)
\]  

(58-1)

\[= PF_n(t) + \xi_n(t) \quad (58-2)\]

with

\[\kappa_n(t) = \sum_{j} x_{n,j} \cdot s_j(t) \quad (58-3)\]

where

- \(PM_n\) = bond \(n\) market price at time \(t\)
- \(PF_n\) = bond \(n\) fitted price at time \(t\)
- \(cf_n(T)\) = bond \(n\) option adjusted cash flow at time \(T\)
- \(PDB(t, T)\) = price at \(t\) of default-free pure discount bond maturing at \(T\)
- \(x_{n,j}\) = bond \(n\) exposure to factor \(j\)
- \(s_j(t)\) = yield spread due to factor \(j\) at time \(t\)
- \(\xi_n(t)\) = bond \(n\) price error at time \(t\)
- \(\kappa_n(t)\) = bond \(n\) total yield spread at time \(t\)

The characteristics of the market as a whole are the term structure, represented here by the default-free pure discount bond prices \(PDB(t, T)\), and the marketwide factor yield spreads \(s_j(t)\). The bond-specific exposures include the option adjusted cash flows \(cf_n(T)\) and the exposures \(x_{n,j}\). These depend upon any call or put options or sinking-fund provisions embedded in bond \(n\). The final bond specific component of this model is the price error \(\xi_n(t)\). This model clearly enumerates how a bond’s total exposure to the various factors determines its price. The estimated values \([PDB(t, T), s_j(t), \xi_n(t)]\) result from filling this model to actual trading prices at time \(t\). All these values change unpredictably over time.

The yield-spread factors \(s_j\) correspond to the non-term-structure sources of risk and return identified by the model. Most of these are sources of default risk. For example, each corporate bond sector might have its own yield spread, measuring the default risk common to all AAA-rated members
of the sector. Each quality rating would also have its own yield spread, measuring the additional default risk common to issues rated lower than AAA.

Beyond the factors that measure default risk, there exist other factors that capture risk and return in bond markets. Benchmark factors measure the uncertain liquidity premiums afforded heavily traded issues. A current-yield factor measures the market’s assessment at time $t$ of the advantage of receiving return in the form of capital gains instead of interest, providing a possible tax advantage. A perpetual factor, appearing in markets containing perpetual bonds, measures the market’s assessment at time $t$ of the advantage or disadvantage of owning perpetual bonds.

Observed corporate bond yield spreads tend to increase with maturity, quantifying the market’s perception of the increase in default risk over time. For investors, any change in the dependence of spreads upon maturity constitutes a source of return risk. Since these spreads appear to increase linearly with duration, a duration spread can measure the extent of this increase with duration at any given time. A risk model can then measure how this dependence changes over time.

So far this analysis had concentrated on the estimated marketwide factors of value. Estimates of these factors rely on option-adjusted cash flows, however. Hence, the next section will describe the option adjustment procedure in more detail.

**Option Adjustments**

Estimating the values $[\text{PDB}(t, T), s_j(t), \xi_n(t)]$ requires both market prices and cash flows and yield-spread factor exposures. However, since embedded options alter the nominal cash flows, the final step in the valuation model involves adjusting the nominal bond cash flows accordingly.

Bonds can include call and put options and sinking-fund provisions. Mortgage-backed securities include prepayment options. These securities are portfolios containing a nonoptionable security and an option. For callable and sinkable bonds and mortgages, the issuer retains the option, and so the portfolio is long a nonoptionable security and short the option:
Optionable bond = Nonoptionable bond - Option  \[ (58-4) \]

and

\[ PF_n(t) = PFN_n(t) - PFO_n(t) \]  \[ (58-5) \]

where

\( PFN_n = \text{bond } n \text{ nominal fitted price} \)

\( PFO_n = \text{bond } n \text{ option fitted price} \)

For putable bonds, the purchaser owns the put option, so the portfolio is long both the nonoptionable security and the option.

Viewed in this portfolio framework, the key aspect of option adjustment involves modeling the embedded option. A detailed description of option modeling is beyond the scope of this chapter, but basically it is a three-step procedure.

First, choose a model that describes the stochastic evolution of future interest rates. This model will describe the drift and, more importantly, the interest rate volatility, of either the short interest rate or the entire term structure. It will describe a set of possible future interest rate paths.

Second, impose a no-arbitrage condition to fairly price bonds of different maturities. This step will determine the probability weight, for valuation purposes, of each possible future interest-rate path and generate a current set of bond prices. A properly tuned model will generate prices consistent with observed bond prices.

Third, impose relevant option decision rules to apply the model to the particular option of interest. These decision rules will depend on both the specific option covenants as well as the behavioral model governing the corporation or the individual mortgage holder. Imposing these rules will lead to estimated cash flows and a price for the option. The portfolio property described in Equation (58-4) then dictates how the option cash flows adjust the optionable bond cash flows.
Option Adjustment Example

To see this work in practice, consider a simple example of a callable zero coupon bond. The bond nominally pays $V$ dollars at maturity $M$:

$$\text{PFN}_n(t) = V \cdot PDB(t, M)$$  \hspace{1cm} (58-6)

But the traded security includes an embedded option for the issuer to call the bond at strike price $K$ and time $T$, with $t < T < M$. The option model estimates the call option value as

$$\text{PFO}_n(t) = -K \cdot Y \cdot PDB(t, T) + V \cdot X \cdot PDB(t, M)$$  \hspace{1cm} (58-7)

where $X$ and $Y$ are cumulative distribution functions. Equation (58-7) resembles the Black-Scholes stock option formula, though $X$ and $Y$ are not necessarily cumulative normal distributions. They do, however, act as probabilities and range between zero and one.

Now consider the interpretation of Equation (58-7): The option involves paying the amount $KY$ at time $T$, to receive $VX$ at the later time $M$. With this interpretation, and with the portfolio property (Equation 58-4), the adjusted price and cash flows for the callable security are

$$\text{PF}_n(t) = V \cdot PDB(t, M) - \left[ -K \cdot Y \cdot PDB(t, T) + V \cdot X \cdot PDB(t, M) \right]$$

$$\text{CPF}_n(T) = K \cdot Y$$

$$\text{cf}_n(M) = V \cdot [1 - X]$$  \hspace{1cm} (58-8)  \hspace{1cm} (58-9)  \hspace{1cm} (58-10)

As Equations (58-9) and (58-10) show, the probabilities $X$ and $Y$ adjust the nominal cash flows. An out-of-the-money option has $X$, $Y$ and PFO all equal to zero, and the option-adjusted cash flows reduce to the nominal cash flows. For this callable bond example, as $X$ and $Y$ increase, the option will shorten the nominal cash flows. More complicated options involve more cash flows (a set of $T_1$, $T_2$, ..., $T_N$), more probabilities, and perhaps even more complicated numerical procedures to estimate the probabilities; but, in principle, the adjustment procedure is the same.

Remember that the true option-adjusted cash flows are still not certain. The option model chooses certain cash flows - $KY$ and $VX$ to replicate both the value and duration of the modeled
security. Unfortunately, it is impossible to choose these cash flows to also replicate the convexity of the modeled security. The discrepancy between the convexity of the modeled security and the convexity of the replicating cash flow—the "excess convexity" of the option—is greatest when the option is at-the-money and approaches zero elsewhere. Fortunately, this discrepancy at worst affects risk modeling only in second order—it affects only convexity, not duration. And, an additional yield-spread factor—an additional $s_j$—can account for the discrepancy.

Given a procedure for estimating these option-adjusted cash flows at time $t$, a set of market prices at time $t$ will lead to estimates of $PDB(t, T)$ and $s_j(t)$, according to a procedure designed to minimize overall pricing error. The historical behavior of these market variables will then lead to the risk model itself.

**THE RISK MODEL**

Bond prices change over time in response to three general phenomena: shortening bond maturities, shifting term structures, and changing yield spreads. Bonds are risky because the last two phenomena are uncertain. The core of a bond risk model is, therefore, an estimate of the variances and covariances of the term structure and the yield-spread factor excess returns. The next two sections describe how to estimate these marketwide factor excess returns, and a third section describes how to estimate bond specific risk.

**Term-Structure Factor Returns**

Building the risk model requires a history of the behavior of all relevant market factors, which the valuation model provides. How exactly does this work? Consider first the term-structure risk factors: the default-free pure discount bond prices. The price $PDB(t, T)$ represents the price at time $t$ of a certain $1.00 paid at time $T$. The return to this factor between $t - \Delta t$ and $t$ is the return to the following strategy:

Invest $1.00 at time $t - \Delta t$ in the default-free pure discount bond $PDB(t - \Delta t, T)$. This bond has a maturity of $T - (t - \Delta t)$. Hold for
a period $\Delta t$. Then sell the bond, now with a maturity $T - t$, for price $PDB(t, T)$.

The excess return to this factor follows by subtracting the risk-free rate of return. This risk-free rate is the return to the strategy:

Invest $1.00$ at time $t - \Delta t$ in the default-free pure discount bond $PDB(t - \Delta t, t)$ maturing at time $t$. This bond has a maturity of $\Delta t$.

Hold for a period $\Delta t$. Then redeem the bond, which has now matured.

The fixed holding period $\Delta t$ is a defining constant of the risk model.

**Yield-Spread Factor Returns**

Now consider the returns associated with the yield-spread factors. The excess return to factor $j$ at time $t$ is the return to the following artificial strategy:

Invest $1.00$ at a time $t - \Delta t$ in a portfolio exposed only to factor $j$ and to term-structure risk. The portfolio duration is set to the average market duration over the risk model history. Hold for a period $\Delta t$, and **roll down the term structure over this period**. Sell the portfolio at time $t$.

This strategy is artificial because it assumes a fixed term structure. The excess return to this strategy is the change in yield spread $s_j$ over the holding period, multiplied by the average bond market duration, plus the yield spread multiplied by the holding period $\Delta t$. Duration, the fractional change in price accompanying a change in yield, enters into this formula to convert a change in yield spread into a price return.

**Specific Return**

Beyond the general, marketwide sources of risk discussed, individual issues also face specific risk. Factors that influence only one particular issue, or only the bonds of a particular company, generate specific risk and return. For example, LBO event risk constitutes a specific risk of current interest. In the context of the risk model, specific returns arise because the bond pricing error $\xi_n(t)$ can change randomly over time. The
specific return to bond \( n \) at time \( t \) is the return to the following strategy:

Invest $1.00 at time \( t - \Delta t \) in a portfolio long bond \( n \), but with all marketwide sources of risk hedged. Hold for a period \( \Delta t \), and then sell. The difference in pricing error will generate the specific return

\[
\xi_n (t) - \xi_n (t - \Delta t) / PM_n (t - \Delta t)
\]

The distinction between marketwide sources of risk and specific risk is important because investors can hedge marketwide sources of risk through other instruments exposed to those same risk sources. By assumption, specific risk is uncorrelated with marketwide risk.\(^8\)

Integration

A multifactor risk model identifies the risk factors operating in a given market and then estimates their risk. Each factor generates excess returns over the model’s estimation period. The risk model analyzes those return histories to forecast their variances and covariances.

Several difficult questions arise during the course of this analysis. What historical estimation period works best for covariance forecasting? Is covariance stable over time, or does it cycle or trend? These basic questions remain the subject of continual debate.

One particular question about forecasting bond market covariance concerns whether or not covariance depends on the level of rates. Does bond market risk increase as rates increase? Is volatility higher when rates are 16 percent than when rates are 8 percent? Academics have speculated that the answer is yes, and historical investigation confirms it, for the U.S. bond market.

John Cox, Jonathan Ingersoll, and Stephen Ross\(^9\) have developed a widely accepted model of the term structure, which prices bonds and bond options based on equilibrium arguments. Their model posits the stochastic evolution of the term structure, with interest-rate standard deviation and bond return standard deviation both proportional to the square root of the level of rates. When rates double from 8 percent to 16
percent, volatility rises by a factor of 1.4: the square root of 2.0.

Historical investigation can probe the dependence of bond market risk on the level of rates. Exhibit 58-1 illustrates the results of a test comparing the standard deviation of monthly pure discount bond excess returns observed each year from 1948 to 1988, to the mean five-year spot rate observed each year. This test determined the exponent $c$ of the relationship

$$\text{volatility} \propto (\text{rate})^c$$

If $c = 1$, then volatility is directly proportional to rates; when rates double, volatility doubles. The Cox, Ingersoll, Ross model assumes that $c = 1/2$. The empirical results illustrated in Exhibit 58-1 demonstrate that $c = 1.08 \pm 0.14$. Within the standard errors shown in Exhibit 58-1, volatility is directly proportional to rate level. Moreover, as the $R^2$ statistic reveals, the level of rates explains 61 percent of the observed difference in risk from year to year. The effect is more pronounced in high-rate periods than in low-rate periods. Further study examined the dependence of yield-spread factor risk on the level of the five-year spot rate. Results were mixed, though generally consistent with direct proportionality.

Given the broad empirical and theoretical evidence supporting the dependence of covariance upon rates, forecasts of covariance based on historical data should take account of this effect.

With all this sophisticated risk model machinery now in place and integrated, how well does the resulting risk model perform?
EXHIBIT 5-1

Risk versus Level of Rates

PERFORMANCE

Multifactor risk modeling involves significant effort. Is this effort justified? Does it significantly differ from the duration approach? How well does the multifactor approach to fixed income risk modeling actually work?

To see how the multifactor approach differs from the duration and convexity approach, consider the performance of a multifactor model in the U.S. bond market. Remember that duration and convexity are both parallel yield shift concepts. They measure the risk of parallel yield shifts. But the term structure does not move in parallel.

The risk model views the term structure as a set of pure discount bonds of different maturities, each allowed to move independently. The covariance matrix then describes the extent to which they actually do move together. Exhibits 58-2 and 58-3 illustrate the two predominant, coherent movements of the term structure, as forecast in September 1989 based on the observed term-structure history throughout the 1980s. These principal components are the independent, uncorrelated collective movements of the term structure. Exhibit 58-2 illustrates the primary term-structure movement: a nonparallel shift, with short rates more volatile than long rates. A duration-based risk model would assume that a parallel shift completely
specified term-structure risk. This nonparallel shift accounts for 95.4 percent of modeled term-structure risk. Exhibit 58-3 illustrates the secondary term-structure movement: a twist, with short and long rates moving in opposite directions. This twist accounts for an additional 4.1 percent of modeled term-structure risk.

EXHIBIT 58-2
First Principal Component

EXHIBIT 58-3
Second Principal Component

To further examine how well multifactor risk modeling performs, the following test compared a simple duration model
and a duration plus convexity model with a 10-factor model (pure discount bonds with maturities of 0.25, 0.5, 1, 2, 3, 4, 5, 6, 7, 10, 30 years) in modeling noncallable U.S. Treasury security returns between January 1980 and October 1986. The noncallable U.S. Treasury market should be the simplest market to model because it requires no factors to account for default risk and no option simulation model. For demonstrating the significant enhancement resulting from the multifactor approach, this is the most difficult test. The results are as follows:

<table>
<thead>
<tr>
<th>Model</th>
<th>Number of Factors</th>
<th>Percent of Explained Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Duration</td>
<td>1</td>
<td>75.8</td>
</tr>
<tr>
<td>Duration + convexity</td>
<td>2</td>
<td>81.1</td>
</tr>
<tr>
<td>First principal component</td>
<td>1</td>
<td>82.4</td>
</tr>
<tr>
<td>First two principal components</td>
<td>2</td>
<td>87.0</td>
</tr>
<tr>
<td>Full multifactor model</td>
<td>10</td>
<td>88.0</td>
</tr>
</tbody>
</table>

The full multifactor model explains significantly more of the observed variance than the simple duration model or even the duration and convexity model. The first two principal components are the optimized first two risk factors. The first principal component model employs just one factor, a nonparallel shift, and outperforms the two-factor duration and convexity model. Of course, one must construct the full multifactor risk model to identify this optimal one-factor model.

This chapter so far has described the construction of a risk model and a test of its overall performance measuring fixed income risk. How, though, does the risk model apply to a particular investment portfolio?

PORTFOLIO RISK CHARACTERIZATION

Historical analysis captures the inherent riskiness of the factors of value present in the bond market. The riskiness of a particular bond portfolio depends upon its exposure to these sources of risk.
The fraction of a portfolio’s present value at each vertex measures the portfolio’s exposure to term-structure risk. Two portfolios with identical distributions of present value along the vertices face identical term-structure risk. Of course, these two portfolios have identical durations. However, two portfolios can have identical durations without having identical distributions across the entire set of vertices. Such portfolios will not face identical term-structure risk.

What about yield-spread factor risk? Consider for example the risk associated with the sector yield spread. The fraction of the portfolio in each sector, multiplied by the duration of the bonds in that sector compared to bond market average duration, measures the portfolio’s sector risk exposure. Risk exposures for quality factors and other factors follow analogously.

Beyond the marketwide factors of value the model identifies, there also exist risk factors associated solely with individual issues. By definition, the specific risk for each issue is uncorrelated with all marketwide factor risk. It may be correlated, though, with the specific risk of other bonds of the same issuer. We can estimate this specific issue risk historically as the realized excess return risk of each specific issue not explained by the model.

Total risk follows from combining the risk exposures that characterize a given portfolio with the variances and covariances of the underlying risk factors that characterize the market, and adding in specific issue risk. This number is the predicted total variance of the portfolio excess return.

Portfolio risk analysis usually involves comparing the portfolio against a benchmark (or liability stream). Comparing risk exposures will quantify the manager’s bets in relation to the benchmark. The risk model can then predict how well the portfolio will track the benchmark. For active managers, an optimizer can implement common factor and specific issue bets, while still controlling risk. An active manager’s utility will usually increase with expected excess return and decrease with expected tracking error. An optimizer can maximize this utility.
SUMMARY

Today’s fixed income markets are characterized by complex instruments and increased volatility. In this environment, bond portfolio management must increasingly rely on sophisticated models to accurately gauge fixed income risk. Building these models requires considerable sophistication. Using them, however, should be straightforward. A good model should simplify the investment process and increase investor insight.

ENDNOTES


4 This section covers more details of the option adjustment process for the benefit of mathematically inclined readers.

5 These cumulative distribution functions correspond to the valuation probability—the martingale probability associated with the stochastic interest-rate model.


The specific risk of two different issues may be correlated, for example, if one company issued them both.