Duration, Convexity and Multiple-factor Models

by Andrew Rudd

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A substantial proportion of pension portfolios worldwide are invested in bonds. In many countries, the bond holding is the largest component, frequently comprising close to 100% of the total portfolio. In the U.S., recent statistics show that bond holdings in corporate pension funds are smaller than equity holdings, but still comprise some 32.7% of the total.¹

This large allocation clearly indicates that sponsors should carefully scrutinize the risk and monitor the returns of their bond positions. Unfortunately, bond portfolios are not as simple to analyze as they once were since managers no longer restrict themselves to holdings in government, agency and high grade corporate issues. Mortgages now comprise approximately 25% of the market and so are commonly held, as are IOs, POs, CMOs and high yield (“junk”) bonds.

These more complex instruments — many of which have embedded option features — have caused considerable problems for both sponsors and money managers. For example, performance measurement, risk management and the sponsor’s oversight function are now all much more complicated. One reason for this is that the traditional tool for analyzing bond portfolios, duration, is less effective in volatile environments and when bonds have embedded options.

One refinement to the traditional approach which employs generalized duration measures (sometimes referred to as a duration vector) has been proposed, but the essential difficulties still remain. A more direct approach — the use of a multiple-factor model — is described here in the context of an immunization strategy. The multiple-factor model is shown to directly solve many of the shortcomings of the traditional approaches.
Duration

A bond’s duration is often used to give an approximation of the price change associated with a given yield change (as shown in Exhibit I). This straight line, or linear, approximation is given by the tangent to the price-yield curve. The error — the difference between the actual price (on the price-yield curve) and the approximate price (on the tangent) — is caused by the curvature (which for many years has been termed convexity in the field of geometry) of the price-yield changes, the accuracy clearly diminishes as yield changes increase. Furthermore, for any fixed move in yield, the greater the curvature in the price-yield relationship, the worse the approximation.

EXHIBIT I

The Approximation Error Inherent in Duration

Consequently, a portfolio manager who only calculates duration can be surprised by bonds with a very positive or negative convexity. For these bonds the straight line approximation — duration — is a particularly inadequate measure of price sensitivity for all but the most infinitesimal yield changes. Unfortunately, this is not the only shortcoming of duration. As is well-known, duration is actually only precisely correct for identical infinitesimal yield changes for all maturities (i.e., for parallel shifts in the yield curve). 2
Therefore, the accuracy of the duration measure:

- is dependent on the actual price-yield curve;
- is a function of the exact pattern of yield curve movements; and
- deteriorates as yield changes become larger

Nevertheless, for some investment applications, where a rough but simple measure of price-sensitivity is required, the use of duration is warranted. However, an important question still remains: can we improve on duration?

**Convexity**

Convexity may be viewed pragmatically as a refinement to duration. While duration implies a constant price response of a bond to yield changes, convexity is a measure of the changing price response as yields rise or fall. It is a measure of the rate at which prices rise as yields fall — and at which prices fall as yields rise. As such, duration and convexity taken together are more representative of the true price response to a given yield change, than is duration alone.

Does convexity solve all the problems with duration? Alas, no. The accuracy of convexity as a measure of the true price response is still dependent on the yield curve dynamics and the characteristics of the particular bond.

**The Story So Far**

The assumptions underlying the use of duration and convexity to monitor and control bond investment strategies can be summarized in two issues:

- The yield-curve moves in parallel shifts. In other words, we can describe the dynamics of the yield curve by the action of a single (bond market) factor, which represents the size of the shift; and
- The differential impact of the bond market factor across a universe of bonds can be attributed to two particular characteristics of each bond; namely, duration and convexity. These attributes capture a bond's exposure to infinitesimal changes in the single factor.

These issues suggest the following important bond management questions:
• Is the bond market factor the only or best factor?

• Are duration and convexity the only bond attributes we should care about? Are duration and convexity systematic and hence priced?

• Is the model easily generalizable to bonds with embedded options, non-Treasuries, etc.?

Before we answer these questions, however, let us turn to the management issues of using duration and convexity with one particular strategy: immunization.

**Immunization**

**With Duration and Convexity**

The objective of an immunization strategy is to protect the net worth of an asset/liability position from interest rate risk so that the present value of the asset is never less than the present value of the liabilities.

The rate of return over a holding period is certain whenever the original investment is placed in a pure (i.e., risk-free) zero coupon bond maturing at the end of the holding period. The discount at the date of purchase will be exactly earned over the holding period yielding an annualized return determined by market conditions on the purchase date.

If the holding period ends before the purchased zero coupon bond matures, then the holding period return becomes subject to *price risk*. The market price at the time of sale may be insufficient to earn the return market conditions originally suggested for this holding period.

Alternatively, if the strategy requires rolling over a maturing bond, then the realized holding period return becomes subject to *reinvestment risk* on the occasion of the sale of the first bond and the subsequent purchase of the second. Only when the pure zero coupon bond is held to a maturity which matches the investment horizon, are price risk and reinvestment risk non-existent. Why? Because the payment at maturity is exactly that promised when the bond was purchased the par value of the bond — and there is but a single payment, so reinvestment risk is absent.

Since pure zero coupon bonds with the appropriate maturity are not always available, how else can you construct an
immunized portfolio? By using a portfolio of coupon bonds to emulate the performance of a hypothetical pure zero coupon bond with maturity equal to the length of the holding period. Coupon bonds may have more than one cash flow over the holding period. So you may think of them as a portfolio of zero coupon bonds with differing maturities. It follows that coupon bonds are individually subject to price risk as well as reinvestment risk.\footnote{5}

Under the assumptions of no spread risk and only parallel shifts in the yield-curve, price and reinvestment risks are exactly offsetting when the immunized portfolio has the same duration as the investment horizon. Hence the price sensitivity of the immunizing portfolio is equal to that of the hypothetical zero coupon bond. This is why conventional immunization strategies have employed a duration matching approach which selects portfolios with a duration equal to the time left in the holding period.

Unfortunately, duration provides a measure of price sensitivity only for a specific type of yield change; namely infinitesimal parallel shifts. Empirically, this assumed form of interest rate uncertainty is quite unrealistic, as shorter term yield changes are more volatile relative to intermediate and longer term yield changes. However, duration may fail to capture the full price response even if a parallel shift in yields were to occur, because the convexity of the immunizing portfolio may not be the same as the zero coupon bond. The use of convexity matching, in addition to duration matching, is an attempt to explain the price sensitivity in a better fashion. And it does. However, also implicit in the convexity measure is the assumption of parallel shifts in yields and, therefore, the duration/convexity matching strategy effectively “locks” in an interest rate assumption.

If the pattern of yield changes turns out to be non-parallel, the strategy fails to achieve the tradeoff between price risk and reinvestment risk. This is because the net price sensitivity of the immunized portfolio to the hypothetical zero coupon is not zero anymore, as it ideally should be. In such a situation, the realized return over the holding pattern may fall below the floor guaranteed at the start of the strategy. Moreover, to the extent that convexity does not capture all of the bond’s curvature, you may fail to meet the immunization target.\footnote{6}

We may summarize the duration/convexity approach as follows:
• If the yield curve moves in infinitesimal parallel shifts, the use of duration is perfectly adequate with homogeneous Government bonds (i.e., there is no spread risk).

• If there are large parallel shifts, duration fails to accurately capture the price sensitivity of the immunizing portfolio; convexity is certainly additionally required, but its use may be inadequate to completely capture the full effects of curvature in the price-yield relationship.

• If there are non-parallel shifts and/or spread risk, duration and convexity are both incorrectly specified; at best they provide only indicative price responses to yield changes.

**Generalized Duration Measures**

One way to correct the deficiencies of the duration/convexity approach is to use generalized duration measures. The simplest development assumes that the yield curve, instead of always being flat and undergoing vertical shifts, has in addition to the shift “factor” and so on. Algebraically, this is equivalent to assuming that the yield curve can be expressed as a polynomial. Straightforward algebra now gives a more complicated formula for the bond price change as a result of infinitesimal shifts, twists, curvature and other changes in the yield curve. In essence, every bond now has its own set of duration-like responses, one for each of the permitted types of change to the yield curve.

The task of immunizing homogeneous Government bonds now requires matching along several dimensions to protect against yield-curve moves of several forms. For example, as discussed above, matching convexity helps prevent a duration-matched portfolio from becoming mismatched as a result of large parallel shifts. Similarly, matching the exposures to the twist factor helps prevent a duration-matched portfolio from becoming mismatched as a result of slope changes in the yield-curve.

The difficulty is that this approach is somewhat non-intuitive (how many of the yield curve movement patterns or factors need to be matched and which are the more important?); and it is difficult to use with bonds with embedded options and other differentiating characteristics. Clearly what is needed is an approach that is not dependent on rigid interest rate assumptions, as well as one
that controls the total investment risk of the strategy in an incisive and comprehensive manner

**The Multiple-Factor Approach**

There are two essential questions underlying the modeling of a bond’s price sensitivity:

- How rich and accurate are the yield curve dynamics?
- Do the chosen bond attributes properly reflect the exposure of the bond to movements in the yield curve?

The multiple-factor approach makes improvements in both areas. In the first area, we would want to introduce additional factors beyond the parallel shift factor to portray twists and oscillations in the yield curve. In this context, the simplest modeling approach is to describe the yield curve dynamics in terms of the returns to pure zero coupon bonds of varying maturity, which represent the yield-curve factors.

Moreover, if we wish to treat more complex bonds, factors which describe changing interest rate volatility (and so influence the valuation of imbedded options) and changing spreads for sectors and lower quality issues will be important.

Further, we need to determine one or more bond attributes which capture each factor’s influence on the bond. These are the factor exposures and are identifiable characteristics of each bond. For example, if there is a factor which causes telephone bonds to perform differentially to industrial bonds (i.e., there exists a variable spread for telephone bonds) then the corresponding factor exposure would be 100% for all telephone bonds and 0% for all others. Alternatively, if there is a liquidity or “seasoning” factor which causes on-the-run bonds to perform differentially to seasoned bonds, then the corresponding factor exposure would be 100% for on-the-run bonds and 0% for all others.
**EXHIBIT II**

**Fixed Income Factors**

<table>
<thead>
<tr>
<th>Category</th>
<th>Exposures</th>
</tr>
</thead>
<tbody>
<tr>
<td>Term Structure</td>
<td>Returns to Pure Discount Bonds of Varying Maturities</td>
</tr>
<tr>
<td>Interest Rate Related Options</td>
<td>Call, Put and Sinking Fund Options</td>
</tr>
<tr>
<td></td>
<td>Conversion and Prepayment Options for Pass-Thrus (Mortgages, CMOs, REMICs, CARS, CARDS)</td>
</tr>
<tr>
<td>Sector and Quality Spreads</td>
<td>Sectors (e.g., Industrials, Telephone, FNMA, etc.) Qualities (e.g., AAA, AA, A, BBB, etc.)</td>
</tr>
<tr>
<td>Tax, Liquidity</td>
<td>Coupon, Issue Size, Treasury Bill</td>
</tr>
</tbody>
</table>

The conceptual similarity between the traditional duration/convexity approach and the multiple factor approach should be clear. The former approach suggests there is only a single factor which causes the yield curve to evolve through time in parallel shifts. The factor exposures are duration and convexity which capture two influences on the bond; the first is a proportional or linear price change while the second is the non-linear (quadratic) impact.

The multiple-factor model utilizes a set of factors which have been found to be significant in explaining the return of a fixed income security. These can be grouped into four broad categories:

- factors describing movements in the term structure;
- factors impacting interest rate-related options;
- factors influencing sector/quality-related spread and default risk; and factors related to tax, liquidity, and payment-convenience effects. Exhibit II gives a list of some of the factors found to be useful in the U.S. markets.

The Treasury yield curve is the key ingredient in the valuation of all fixed income securities. It serves as the “jumping off”
point for valuation of any category and pattern of cash flows. Interest rate related options include any embedded call or put option, and the value of the sinking fund provision which amounts to a partial call option. Important differences in values of non-Treasury securities are related to perceived or actual differences in quality. The natural definition for quality is in terms of the probability of default and the resultant loss of principal or interest. In addition, lower quality is frequently related to lower marketability: during times of tight money, “lower quality” securities may be difficult to trade.

The multiple-factor risk model provides a ranking of the factors from most to least variable, the correlations between them, as well as the magnitude of the “specific risk” of individual bonds — namely that risk which is unique to the bond and not explained by the influences of the factors. The risk for an individual bond or bond portfolio may be calculated both as the risk of total return and tracking error relative to a benchmark. Tracking error is a measure of the potential deviation of the portfolio return from that of the benchmark.

**Immunization Revisited**

One benefit of the multiple-factor approach to portfolio optimization is the reduction of tracking error. Intuitively, we can see it performing the task by trying to balance the risk exposure of each factor in the benchmark, against risk exposure of the same factor in the portfolio.

It may not be possible to match all of the factor exposures exactly. In this case, we certainly try to match the most important and there by take into account the correlations among the factors so that an overexposure on one may compensate for an underexposure on another. In addition to the mismatched factor exposures, there may also exist factors which are not relevant to the benchmark but are inescapably present in the universe of bonds. In this situation, the portfolio will inevitably be exposed to additional “active” risk relative to the benchmark.

In the context of an immunization strategy, the benchmark is the hypothetical zero coupon bond with maturity equal to the length of the holding period. The multiple factor approach chooses a portfolio such that the portfolio factor exposures relative to those of the hypothetical zero coupon bond results in the residual risk being minimized. This directly minimizes the possibility of differential performance between the hypothetical zero coupon bond and portfolio.
Thus, unexpected changes in the term structure of any pattern should have minimal effect on the value of the portfolio relative to that of the zero coupon bond.

Another way of viewing the exercise is that the multiple factor optimization model emulates the zero coupon bond by taking into account all the important “dimensions” in the bond market. It is therefore a more general and complete approach than looking only at duration and convexity. It is also more intuitive than the generalized duration approach since the factors are easily identifiable and hence can be directly related to the various distinguishing features of the bonds.

**Global Bond Portfolios**

The multiple-factor approach can be directly applied to the management of global bond portfolios. For example, a national multiple factor model could be estimated within each country. These models would then be integrated through the use of another model to capture the correlations between the national factors, currency movements and, perhaps, global macroeconomic influences.

Two benefits of the approach are that the impact of currencies (e.g., whether the strategy is hedged or unhedged) is directly obvious and that any insights from each national market are immediately captured at the global level. In contrast, the duration/convexity approach is quite obscure in the global context because there is no single interest rate process; instead there are multiple processes (one for each country) connected by the various currency movements.

**Conclusion**

The pension sponsor and portfolio manager both need to be concerned with the risk levels and price sensitivity of their portfolios. The potential for mis-specification of price sensitivity, as well as the piecemeal nature of the approach results in the incompleteness of the conventional duration measure. Knowing the duration of a portfolio does give some information, but it is a potentially misleading measure of its riskiness. Adjusting the cash flows, and hence duration, for the effect of embedded options gives a more refined measure of the portfolio’s price sensitivity. Typically, however, this measure will not reflect any sector or quality risk.
In contrast, the multiple-factor approach, due to its comprehensive nature, results in a more effective strategy implementation. At the least, the sponsor or portfolio manager should be aware of the portfolio’s risk relative to a well known market index. The relevant measures are the portfolio beta and tracking errors — both of which can be readily attained from multiple-factor model computations. Moreover, these measures will explicitly take account of nonhomogeneities of the constituent bonds and non-parallel yield curve dynamics.

Finally, in a multiple manager context, where the sponsor's total portfolio may be apportioned among several bond managers, the sponsor should be aware of the correlations between the various components. In this way, a judgement can be made as to whether the managers are following sufficiently different styles with sufficient aggressiveness for each to be providing value-added. This type of analysis, all but impossible in a duration/convexity framework, is straightforward in the multiple-factor model environment.

1 “Table 6A- Portfolio Mix of Combined Corporate Pension Funds,” Money Market Directory of Pension Funds and Their Investment Managers, 1988, p. xvii.

2 It is possible to define duration-like measures for particular non-parallel shifts. However, each of these measures is accurate only for the hypothesized yield-curve dynamics.

3 There are a multitude of definitions for convexity as there are for duration. Unfortunately, they are all dependent on the yield-curve dynamics.

4 Technically, convexity is the second order (or quadratic) approximation to the price yield curve just as duration is the first order or linear approximation. Hence, if the price-yield curve is not quadratic, duration and convexity combined will not capture all its curvature.

5 They may also be exposed to other types of risk, such as sector and quality risk, and changing call option values. Moreover, in some countries, even Government bonds with particularly high or low coupons may be exposed to some risk as a result of a tax effect. These risks are related to “spreads” associated with differentiating characteristics of the bonds, and will be considered below.
It is possible to compute an upper bound on the error for risk free assets in following the traditional “duration matching” immunization strategy. This upper bound depends in part on the dispersion of the immunizing portfolios cash flows relative to the hypothetical zero coupon. See Gifford Fong and Oldrich Vasicck, “Risk Minimizing Strategy for Multiple Liability Immunization.” Working Paper, September 1980.


Interestingly, convexity is similar to a risk-free bond’s exposure to the twist factor so convexity arises in two different bond contexts.

This multiple-factor approach can easily accommodate non-Treasuries since classes of non-homogeneous bonds will be exposed to additional factors. The duration/convexity approach is less amenable to such generalizations since both duration and convexity will have to be adjusted for the resulting influence of these factors. This frequently leads to inconsistencies; for example, accurately adjusting duration for the embedded options requires the assumption of yield curve dynamics other than parallel shifts.