Pricing Methodology of Forward Rate Agreements (FRAs)
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Forward Rate Agreements or FRAs were first introduced in 1984, and by the end of that year an inter-bank market was established in London and Continental Europe, reaching the United States shortly thereafter. A simple OTC hedging product, the FRA is the building block for all forward LIBOR-based interest rate curves. Although the FRA is comparatively old among derivative instruments, it continues to broaden its appeal among corporate and financial users, because it is not credit intensive and it lends itself well to customization.

Contract Description

An FRA is a cash settled contract between two parties where the payout is linked to the future level of a specified reference interest rate, such as 1, 3 or 6 month LIBOR. The two parties fix an interest rate to be paid on a hypothetical deposit that is initiated at a specific future date, known as the settlement date.

At the settlement date, the buyer of the FRA commits to pay interest on this hypothetical deposit at the agreed rate and receives interest at the prevailing value of the reference rate. The seller takes the reverse position.

The two parties exchange the net payment at the settlement date. The actual cash flow involved is the present value of the net difference, where the prevailing value of the reference rate is used as the discount rate.

Example 1

A bank buys a “threes sixes” or 3 x 6 FRA (“deposit” starts in three months and ends in 6 months) with a notional or face value of $100 million.

Let the FRA be agreed at 4%. Three months later the prevailing rate for 3-month LIBOR is 6% and the deposit period extends for 92 days on a payment basis of actual/360.
Clearly, the bank will receive a flow of funds which is determined as follows:

\[
\begin{align*}
&= \left(100 \text{M} \cdot \left(6\% - 4\% \right) \cdot 92 / 360 \right) / \left(1 + 6\% \cdot 92 / 360 \right) \\
&= \left(511,111.1111 / \left(1 + 6\% \cdot 92 / 360 \right) \right) \\
&= \$503,392.43
\end{align*}
\]

Exhibit 1: Cash flows for a 3 x 6 “threes sixes” FRA with the two payment scenarios

Pricing FRA Contracts Using Interest Rate Futures

Interest rate futures are often used to hedge and price FRA contracts. For the purposes of this document it will be assumed that there is no difference between futures and forwards contracts. In practice, FRA rates will be lower than those implied by the futures rates because of the convexity difference between the two instruments. Most market participants, however, agree that when the FRA maturity is under a year or so, the magnitude of the discrepancy — less than a basis point — is thought to be acceptable.¹

In this section, we will review how to:

- Express FRAs in terms of discount bond prices.
- Calculate FRA strip rates from a yield curve comprised of a cash stub rate and interest rate futures contracts.
- Incorporate year end turn affects into a futures yield curve.

¹ See Appendix: The Convexity Correction for Futures Contracts.
FRAs In Terms of Discount Bonds

Using simple interest calculations, we can determine the value to which $1 would grow if invested at a given rate for a period of time. It is of course important to assume that the period is less than a year. The resulting sum of money — principal plus interest — is referred to as the terminal wealth \((TW)\). The inverse of the \(TW\) (based on a principal of $1) is known as the zero-coupon or discount bond price \((P)\) for that future date.

Example 2

Let us work out the terminal wealth and discount bond price based on $1 being invested for 92 days or 3 months at 6.00% (the rate is a simple annual rate based on a 360-day year).

\[
TW(0,3) = 1.0153333
\]

Given that the discount bond price is the inverse of the \(TW\),

\[
P(0,3) = 0.9848982
\]

If the yield curve were flat, and the six month rate was also at 6.00% for a period of 182 days on the same day basis, using the approach demonstrated above, one finds that the terminal wealth and discount bond prices at six months are:

\[
TW(0,6) = 1.0303333
\]
\[
P(0,6) = 0.9705597
\]
Exhibit 2: The terminal wealth and discount bond prices at three and six months given that interest rates are flat at 6% for both periods — based on simple annual rates and an actual/360 day basis.

The rate for the 3 x 6 FRA can be determined using the above values. To eliminate the opportunity for arbitrage, the terminal wealth at six months from $1 must be the same as terminal wealth derived from investing $1 in the three month deposit and taking the proceeds and investing them in a 3 x 6 FRA.

To summarize:

\[
TW(0, 6) = TW(0, 3) \cdot TW(3, 6).
\]

Likewise,

\[
P(0, 6) = P(0, 3) \cdot P(3, 6).
\]

Re-arranging the above gives us:

\[
P(3, 6) = P(0, 6)/P(0, 3).
\]

\(3 \times 6\) FRA or \(FRA(3, 6)\) can be determined as follows:

\[
FRA(3, 6) = \left(\frac{1.0}{P(3, 6)} - 1.0\right) \cdot \text{Year basis/Days (3, 6)}
\]

The generic case is summarized below:

\[
FRA(t, T) = \left(\frac{P(0, t)}{P(0, T)} - 1.0\right) \cdot \frac{Y_{\text{basis}}}{D_{(t, T)}}
\]
Notes

1. $P(0, T)$ is the price at time $T$ of a discount bond that pays off $1 at time $T$.

2. $Y_{basis}$ is the year basis
   
   e.g., 360 for Actual/360 (USD, DEM, JPY) and 365 for Actual/365 (GBP).

3. Days $(0, T)$ is the number of days from time $T$ and $T > t$.

Example 3

From the previous example:

$P(0, 3) = 0.9848982$, $P(0, 6) = 0.9705597$.

Days $(3, 6) = 90$ and $Y_{basis} = 360.0$, using equation (1) gives:

$$FRA(0, 3) = \left( \frac{P(0, 3)}{P(0, 6)} - 1.0 \right) \cdot \left( \frac{Y_{basis}}{Days(3, 6)} \right)$$

$$= \left( \frac{0.9848982}{0.9705597} - 1.0 \right) \cdot \left( \frac{360.0}{90.0} \right)$$

$$= 0.0590939$$

$$FRA(0, 3) = 5.90939\%$$

It may seem strange that the forward rate over the period from 3 to 6 months in this example differs from the rate from 0 to 3 months, even though we assumed that the (simple) interest rate was constant over the entire time. The reason is just that the simple interest rate is not compounded, while the rollover “replicating” strategy using FRAs gains through compounding. In fact, the rate for the 3 x 6 FRA is just the simple interest rate discounted by $P(0, 3)$:

$6\% \cdot P(0, 3) = 5.90939\%$

Pricing Discount Bonds Using Interest Rate Futures

From equation (1) it is clear that we need discount bond prices to calculate FRA strip rates. We can use methods that are established as practice in the money markets, to determine the value to which $1 would grow if invested in a sequence of cash and Eurodollar futures (implied) interest rates.
Example 4

Let us determine the terminal wealth ($TW$) and the price of the discount bond for the value date of the March contract, which is 89 days from value Spot (12/20/96):

$$TW\left(\text{spot, 19 Mar 97}\right) = S \cdot \left(1 + 5.58\% \cdot 89/360\right)$$

$$TW\left(\text{spot, 19 Mar 97}\right) = 1.0137950$$

$$P\left(\text{spot, 19 Mar 97}\right) = 0.9863927$$

Let us now extend the horizon and consider the terminal wealth value and discount bond price for the value date of the June contract:
The terminal wealth values and discount bond prices for the remaining futures contracts can be worked out in a similar fashion and the results are summarized below:

### Exhibit 4: Summary of terminal wealth values for 3 month Eurodollar futures contracts as of 12/18/96

<table>
<thead>
<tr>
<th>Contract</th>
<th>Value Date</th>
<th>Rate</th>
<th>Terminal Wealth</th>
<th>Discount Bond Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stub</td>
<td>12/20/96</td>
<td>5.58</td>
<td>1.0000000</td>
<td>1.0000000</td>
</tr>
<tr>
<td>Mar 97</td>
<td>03/19/97</td>
<td>5.58</td>
<td>1.1037950</td>
<td>0.9863927</td>
</tr>
<tr>
<td>Jun 97</td>
<td>06/18/97</td>
<td>5.74</td>
<td>1.0280946</td>
<td>0.9726732</td>
</tr>
<tr>
<td>Sep 97</td>
<td>09/17/97</td>
<td>5.88</td>
<td>1.0430117</td>
<td>0.9587621</td>
</tr>
<tr>
<td>Dec 97</td>
<td>12/17/97</td>
<td>6.06</td>
<td>1.0585143</td>
<td>0.9447204</td>
</tr>
</tbody>
</table>

To determine the discount bond price for a given date that lies between two futures value dates, sensible values can be obtained by interpolation using the following equation: 

\[
TW(0, T_2) = TW(0, T_1) \cdot TW(T_1, T_3)^{(T_2 - T_1)/(T_3 - T_1)}
\]

### Notes
1. \(T_1\) and \(T_3\) are successive futures value dates.
2. \(T_2\) lies between \(T_1\) and \(T_3\).

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2 See Gelen Burhardt, Bill Hoskins, Susan Kirshner, "Measuring and Trading Term TED Spreads," Research Note published by Dean Witter Institutional Futures, July 26, 1995, Appendix 10–15. Implicitly, this formula assumes that it is the continuously compounded interest rate that is constant from \(T_1\) to \(T_3\), not the simple interest rate.
Example 5

Let us work out the value for a 3 x 6 (3/20/97–6/20/97) USD FRA based on LIBOR. In order to accomplish this, we need discount bond prices for these dates.

**Step 1: determine discount bond price for 3/20/97**

Using data from Exhibit 3:
\[
\begin{align*}
TW(20 \text{ Dec} 96, 19 \text{ Mar} 97) &= 1.0137950 \\
TW(19 \text{ Mar} 97, 18 \text{ Jun} 97) &= 1.0141050
\end{align*}
\]

March 20th is a valid business day and overlaps the March futures value date by a single day; the gap between the March 97 and June 97 futures value dates is 91 days.

Using equation (3):
\[
\begin{align*}
TW(20 \text{ Dec} 96, 20 \text{ Mar} 97) &= 1.0137950 \cdot 1.0141050^{(1/91)} \\
TW(20 \text{ Dec} 96, 20 \text{ Mar} 97) &= 1.0139511 \\
\dot{P}(20 \text{ Dec} 96, 20 \text{ Mar} 97) &= 0.9862409
\end{align*}
\]

**Step 2: determine discount bond price for 6/20/97**

Using data from Exhibit 3:
\[
\begin{align*}
TW(20 \text{ Dec} 96, 18 \text{ Jun} 97) &= 1.0280946 \\
TW(19 \text{ Mar} 97, 18 \text{ Jun} 97) &= 1.0145094
\end{align*}
\]

June 20th is a valid business day and overlaps the June futures value date by two days; the gap between the June 97 and September 97 futures value dates is 91 days.

Using equation (3):
\[
\begin{align*}
TW(20 \text{ Dec} 96, 20 \text{ Jun} 97) &= 1.0280946 \cdot 1.0145094^{(2/91)} \\
TW(20 \text{ Dec} 96, 20 \text{ Jun} 97) &= 1.0284201 \\
\dot{P}(20 \text{ Dec} 96, 20 \text{ Jun} 97) &= 0.9723653
\end{align*}
\]
**Step3: calculate 3 x 6 FRA (3/20/97,6/20/97) strip rate**

There are 92 days in the above period, and the year basis is actual 360.

Using equation (1):

\[
\text{FRA}(3/20/97, 6/20/97) = \left( \frac{0.9862409}{0.9723653} - 1.0 \right) \cdot 360/92 \\
= 0.0558391 \\
\text{FRA strip}(3/20/97, 6/20/97) = 5.58391\%
\]

It should be remembered that the implied FRA strip rate from the futures data is not identical to the expected FRA rate, because futures instruments do not exhibit convexity and are marked to market.

**Year End Turn Analysis**

The turn is the financing period from the last business day of one calendar year to the first business day of the next. For example, the turn for 1996/7 for US Dollars ran for two days: December 31st was the last business day of 1996, January 2nd was the first business day of 1997.

Historically, at the year end there is a scarcity of funds and short term interest rates have been known to dramatically rise for the period of the turn. As trading commences in the new year, rates tend to return to prior levels. This is known as turn pressure and will affect any interest rate instrument that spans the year end, such as December interest rate futures contracts.³

Calculating FRA strips from a yield curve comprised of futures that have not been adjusted for the year end turn effect leads to misleading values. For a futures contract that spans the year end, the \( (1.0 + F_{i} \cdot \text{Days} \cdot \frac{F_{i+1}}{Y_{\text{basis}}}) \) term used in equation (2) is substituted by the right-hand side of equation (4).

\[
TW\left(F_{i}, F_{i+1}\right) = \left(1 + \frac{r}{Y_{\text{basis}}} \right) \cdot \left(1 + \frac{r_{\text{turn}} \cdot \text{days in turn}}{Y_{\text{basis}}} \right) \cdot \left(1 + \frac{b}{Y_{\text{basis}}} \right).
\]

Notes

1. \( TW\left(F_{i}, F_{i+1}\right) \) is the TW between two successive futures contracts \( F_{i} \) and \( F_{i+1} \).
2. \( r_{\text{turn}} \) is the elevated rate over the year-end turn period.
3. \( r \) represents the normalized rate for the period outside the year-end turn.
4. \( F_{i} \) is a futures contract straddling the year-end turn.
5. \( a \) represents the number of days between the start of \( F_{i} \) and the year-end turn.
6. \( b \) represents the number of days between the year-end turn and the maturity of \( F_{i+1} \).

Note that this formula is based on a constant simple interest rate on non-turn dates over the period, so it is slightly inconsistent with equation (3). However,
for reasonable levels of interest rates (e.g., under 20%), the discrepancy between the two methods is far less than 1 bp for calculation of the terminal wealth over any subperiod of the contract spanning the year end.

Appendix: The Convexity Correction for Futures Contracts

In calculating the forward curve from futures prices, we have assumed that a futures contract is priced as though it were a forward contract on the underlying. Because of interest rate volatility, this assumption is not perfectly accurate. Due to the effect of daily resettlement, the price of a Eurodollar futures contract should generally be less than that of a corresponding forward contract. This is because a long position in the futures leads to cash inflows when rates drop and outflows when rates rise. So cash must be invested after rates fall and borrowed after rates rise. The negative impact of these reinvestments and borrowings must be compensated for by a lower initial contract price than would be obtained for a forward contract. The effect is proportional to the variance (squared volatility) of interest rates: the size of the resettlement amounts is proportional to the volatility, and so is the differential between the lending and borrowing rates applicable to these cashflows.

A simple argument gives the main contribution to this “adjustment” over long time scales. We can calculate the typical cost of the reinvestment and borrowing of daily resettlements as follows. The size of the average fluctuation over a short time \( \Delta t \) (e.g., 1 day) is \( \pm \sigma y \Delta t \) where \( y \) is the interest rate and \( \sigma \) is the volatility. This produces a resettlement cashflow of \( \pm \tau \sigma y \sqrt{\Delta t} \) where \( \tau \) is the contract term (e.g., 0.25 year). The average time for which such a payment or debit must be invested or borrowed is \( T/2 \), where \( T \) is the time to contract expiration. There are \( \frac{T}{\Delta t} \) of these cashflows over the contract life. The contract price is lowered by the sum of these average costs, giving a price adjustment for the futures contract of \( -\tau \sigma^2 y^2 T^2 / 2 \). This translates into an adjustment to the interest rate implied by the futures contract of \( -\sigma^2 y^2 T^2 / 2 \).

The precise value of this correction is model dependent: one needs to make assumptions regarding how the forward curve moves around over time and
about the mean reversion of rate movements. However, any reasonable model with similar interest rate volatility will give similar results on time scales of up to a few years.

In the modified Vasicek model (also known as the Hull-White or mean reverting Gaussian model), and for times short enough so that mean reversion doesn’t have a substantial impact, the correction is equivalent to an downward shift of the forward curve by $\sigma^2 y^2 \left( T^2 / 2 + T \tau \right)$. Due to mean reversion, this correction doesn’t grow without bound. Instead, the value of $T$ in the formula is effectively capped by $1/\kappa$ where $\kappa$ is the mean reversion. Empirically, this value is of order 0.03 in the US, so even on time scales as long as 10 years the mean reversion doesn’t play a significant role in this calculation.

A reasonable “ballpark” assumption for the volatility $\sigma$ is about 20% when the short rate is 5%. Then the convexity correction is about $0.5 \text{ bp} \cdot (\text{time})^2$ (slightly more for times under 1 year). Since this is an upward bias to the forward curve implied by the futures price, it must be subtracted from the implied rate to get the convexity adjusted forward curve.