Efficient Replication of Factor Returns: Theory and Applications

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The empirical evidence of Chen, Roll, and Ross [1986] and Fama and French [1993] suggest that asset dynamics are best characterized by a multifactor representation of asset returns. As in Sharpe [1963], beta now captures multiple sources of systematic risk; exposures to each systematic risk are compensated by their respective risk premia. Passive investment management essentially aims to optimize exposures to various sources of beta. Active investment management aims to optimize sources of alpha that are unrelated to systematic beta risks. In either context investors can capture the premium or hedge the risk associated with a particular beta factor through factor-mimicking portfolios. These portfolios have unit exposure to a target factor, zero exposure to other factors, and minimum portfolio risk. The ability to transact in factor-mimicking portfolios may thus provide an efficient enhancement to the practice of investment management.

In this article, we examine different methods for constructing factor-mimicking portfolios. In particular, given a factor-based model of risk and return, we consider two types of factor replication. Full replication uses the underlying information embedded in the model to re-engineer the targeted factor return, and optimized replication uses constrained mean–variance optimization with appropriately chosen expected asset returns and asset covariance matrix. In addition to imposing factor exposure constraints, we examine methods that impose limits on turnover, asset holdings, and the number of assets in the portfolio in order to make factor portfolios easier to implement in practice. This analysis is applied in the context of the Barra Global Equity Model (GEM). We present evidence that the momentum and value factors can be efficiently replicated. Our tracking portfolios capture the risk and return properties of these factors for the period analyzed (January 1998–June 2008) as well as for periods of extreme market turbulence.

Portfolios that efficiently replicate factor returns can be utilized to potentially enhance the risk–and-return profile of both passive and active investment strategies, as illustrated by the evidence we present. We show that a manager who had attempted to track the MSCI World Small Cap Index could have enhanced the risk–return profile of this passive strategy by hedging the factor exposures unrelated to size over the period December 2003 to December 2008. We also consider a perfect-foresight active stock-picking strategy that is long the 50 top-performing U.S. stocks and short the 50 bottom-performing U.S. stocks. In this context, factor-mimicking portfolios can be utilized to hedge out the unintended factor exposures, leaving the manager with pure stock-selection alpha. Using the Barra U.S. Equity Risk (USE3) model, we illustrate...
how the risk-and-return profile of this active investment strategy could have been enhanced by neutralizing some of the large factor exposures over the period December 1997 to June 2008.

This article is presented in two sections. We first provide analytical considerations in the construction of factor-mimicking portfolios along with empirical evidence. We subsequently illustrate both passive and active management implementation of factor-mimicking portfolios.

FACTOR-MIMICKING PORTFOLIOS: ANALYTICS

In this section, we provide analytical considerations in the construction of factor-mimicking portfolios as well as empirical evidence.

Full Replication

We consider the problem of constructing factor-mimicking portfolios corresponding to the factors of a fundamental multifactor model described by the following equations:

\[ r = Xf + e \] (1)
\[ V = XFX' + D \] (2)

These equations decompose return and risk into a systematic component (respectively \( Xf \) and \( XFX' \)) and a specific component (\( e \) and \( D \), respectively). Factor models are often estimated through weighted cross-sectional regression. In this regression, observations are typically weighted by the square root of market cap or by the inverse of specific volatility. This is done to ensure that the estimated factor returns are not unduly influenced by very small or very volatile assets. In this case, we can directly compute the weights of factor-mimicking portfolios from the factor return estimation regression in which \( W \) represents the weight applied in the ordinary least squares (OLS) estimation:

\[ r = Xf + e \Rightarrow f = (X'WX)^{-1} X'Wr \] (3)

Equation (3) computes factor returns \( f \) as a weighted average of asset returns \( r \). In this equation, the rows of matrix \((X'WX)^{-1}X'W\) correspond to the weights of the factor-mimicking portfolios. The main advantage of this method is that the resulting portfolios replicate exactly the factor returns estimated by the multifactor model.

However, an important drawback of this method is that it does not necessarily lead to factor-mimicking portfolios with minimum ex ante risk; in practice, this is an important consideration as the error term may represent systematic risks that are not captured by the empirical factor model. Furthermore, the resulting portfolios have long or short positions in all the assets in the estimation universe and these positions can change significantly from one period to the next. Therefore, these theoretical portfolios that replicate exactly the factor returns from multifactor models may be difficult or costly to implement in practice.

Optimized Replication

We can specify the factor-mimicking portfolio construction problem in general terms as follows. Given a factor model, we would like to construct portfolios that have maximum exposure to a target factor, zero exposure to all other factors, and minimum portfolio risk. We can express this problem as a general mean-variance optimization problem,

\[ \max_{h} \left\{ h'X_{\alpha} - \frac{1}{2} \lambda h'Vh \right\} \] (4)

\[ s.t. \quad h'X_{\sigma} = 0 \] (5)

Here \( X_{\alpha} \) and \( X_{\sigma} \) represent exposures to the target factor and to all other factors. This constrained optimization problem can be solved analytically using the method of Lagrange multipliers. Optimal portfolio weights are given by the following expression:

\[ h^* = \frac{1}{\lambda} V^{-1} [X_{\alpha} - X_{\alpha}(X_{\alpha}'V^{-1}X_{\alpha})^{-1}(X_{\alpha}'V^{-1}X_{\alpha})] \] (6)

One interesting observation is that if we substitute \( V^{-1} \) in Equation (6) with the weights \( W \) used in the factor return estimation regression, then the factor portfolios we obtain through Equation (6) are exactly the same as the full replication portfolios obtained through Equation (3). Therefore, the full replication method can be viewed as a special case of the general optimization framework expressed in Equations (4)–(6). Another interesting observation is that, as we maximize exposure to a target factor and constrain exposure to all other risk factors, we are effectively minimizing specific risk; therefore, we could substitute the...
total risk matrix \( V \) with the specific risk matrix \( D \) in Equation (4). In the appendix, we show that using specific risk instead of total risk in the optimization leads to the same optimal portfolios up to a scaling parameter.

In general, optimized replication leads to portfolios that replicate closely, but not exactly, the factor return estimated by the model. In the special case where the inverses of specific variances were used as weights in the model cross-sectional regression, the optimized replication method exactly captures the factor return estimated by the model and leads to the same factor-mimicking portfolios as the full replication method.

**Adding Investability Constraints**

Full replication and optimized replication portfolios have long or short positions in all the assets in the underlying model estimation universe. Liquidity considerations or the availability of shorts may impact the ability to implement such a strategy in practice. In addition, positions can change significantly from one period to the next, possibly resulting in high portfolio turnover and significant transactions costs.

In order to make factor portfolios easier to implement and manage, investors may wish to impose additional constraints on these portfolios. For example, internal risk management controls or regulatory requirements may impose limits on the leverage of the long–short factor-replicating portfolio. Similar institutional requirements may impose constraints on the amount of capital that can be allocated to a single asset or a group of assets in the portfolio. Also, low liquidity, limited borrowing availability, increased operational complexity, and transaction cost considerations make it cumbersome to manage portfolios that require long or short positions across many assets, especially medium and small capitalization assets. As a result, investors may wish to impose constraints on the turnover, maximum asset weight, and number of assets in the factor-replicating portfolio. Our general optimization framework for constructing factor-mimicking portfolios can be extended to include different types of constraints and benchmarks.

**Constructing Factor Portfolios Using Active Risk Optimization**

So far, we have expressed the factor replication problem as a total risk optimization problem,

\[
\max_h \left\{ h'X_a - \frac{1}{2} \lambda h'Vh \right\}
\]

(7)

In certain applications, however, minimizing tracking error relative to the target factor return is more important than minimizing the total risk of the factor portfolio. In this case, using \( h_f \) to represent the full replication portfolio weights, we can express the factor replication problem as an active risk optimization problem,

\[
\max_h \left\{ h'X_a - \frac{1}{2} \lambda (h - h_f)'V(h - h_f) \right\}
\]

(8)

These two different optimization approaches for constructing factor portfolios (total risk optimization and active risk optimization) may be appropriate for different applications. In general, investors can use factor portfolios to capture systematic returns or to hedge common factor risks. In the former application, investors wishing to capture risk premia may place more emphasis on minimizing the total risk of the factor portfolio, in which case the total risk optimization approach would be more appropriate. But for investors using factor portfolios to hedge their exposure to common factor risk, the active risk approach may be more appropriate, as it ensures closer tracking and, therefore, better hedging of the targeted factor.

**Empirical Results**

We use data from the Barra Global Equity Model (GEM) to analyze five different factor replication methods. First, we compare unconstrained full replication and optimized replication portfolios. Then, we examine three constrained replication methods with constraints on turnover and on the maximum number of assets in the replicating portfolio.

We test these five replication methods over the period January 1998 to June 2008. At the beginning of each month we form long–short portfolios that target the value factor and the momentum factor for each method. The weights of the full replication and optimized replication portfolios are computed using the analytical formulas presented in the full replication and optimized replication sections of the article. The weights of the other replication portfolios are computed numerically. The exposures and variance-covariance forecasts that are used to construct the factor portfolios are given at the beginning of
each month and are based solely on past data available at the beginning of each month.

Exhibit 1 presents the performance of the five replicating portfolios. On the one hand, total risk methods achieved lower predicted, as well as realized, portfolio risk, leading to a higher realized Sharpe ratio. For example, the value portfolios based on total risk optimization had realized volatility of 1.97% and 1.98% compared with realized volatility of 2.42% and 2.45% for the value portfolios based on active risk optimization. On the other hand, active risk methods experienced lower realized tracking error than total risk methods. For example, momentum portfolios based on total risk optimization experienced tracking errors of 1.53% and 1.62% compared with tracking errors of 0.86% and 1.07% for momentum portfolios based on active risk optimization. Momentum portfolios were generally more volatile and had higher tracking error and higher turnover compared to value portfolios. Specifically, momentum portfolios had average monthly one-way turnover of around 24% compared to around 15% for value portfolios based on the same methods. Nonetheless, even when we allow for a very conservative transaction cost estimate of 25 basis points (bps), the replicating portfolios with constrained turnover broadly succeed in capturing the risk-adjusted performance (net of costs) of the target factor.

Though the first two sample moments of the target factor appear to be captured by our factor-mimicking portfolios, it is also important to examine tracking performance during periods of high volatility or extreme market conditions. This analysis is particularly relevant when the portfolios are utilized as hedging instruments. Exhibit 2 reports the performance of the five replication methods during periods of extreme market conditions. The results highlight that the hedging portfolios, especially those based on active risk optimization, track factor returns relatively well even during the observed periods of extreme market turmoil.9

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**Exhibit 1**

Performance of Five Replicating Portfolios

<table>
<thead>
<tr>
<th>Portfolio Statistics</th>
<th>Full Replication</th>
<th>Total Risk Optimization</th>
<th>Active Risk Optimization 10% Turnover</th>
<th>Active Risk Optimization 10% Turnover</th>
<th>Active Risk Optimization 10% Turnover 400 Assets</th>
</tr>
</thead>
<tbody>
<tr>
<td>GEM Value Factor Portfolios</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average Monthly Return</td>
<td>0.25%</td>
<td>0.27%</td>
<td>0.25%</td>
<td>0.21%</td>
<td>0.21%</td>
</tr>
<tr>
<td>Annualised Total Risk</td>
<td>2.46%</td>
<td>1.98%</td>
<td>1.97%</td>
<td>2.42%</td>
<td>2.45%</td>
</tr>
<tr>
<td>Realised Sharpe Ratio</td>
<td>1.23%</td>
<td>1.667%</td>
<td>1.546%</td>
<td>1.039%</td>
<td>1.029%</td>
</tr>
<tr>
<td>Annualised Tracking Error*</td>
<td>0.00%</td>
<td>1.12%</td>
<td>1.13%</td>
<td>0.33%</td>
<td>0.73%</td>
</tr>
<tr>
<td>Average Specific Risk Forecast</td>
<td>1.17%</td>
<td>0.82%</td>
<td>0.84%</td>
<td>1.18%</td>
<td>1.31%</td>
</tr>
<tr>
<td>Average Total Risk Forecast</td>
<td>2.53%</td>
<td>2.38%</td>
<td>2.38%</td>
<td>2.53%</td>
<td>2.59%</td>
</tr>
<tr>
<td>Portfolio Leverage</td>
<td>111.0%</td>
<td>106.1%</td>
<td>107.9%</td>
<td>112.0%</td>
<td>89.7%</td>
</tr>
<tr>
<td>Monthly One-Way Turnover</td>
<td>15.2%</td>
<td>14.9%</td>
<td>9.9%</td>
<td>8.9%</td>
<td>10.1%</td>
</tr>
<tr>
<td>GEM Momentum Factor Portfolios</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average Monthly Return</td>
<td>0.08%</td>
<td>0.05%</td>
<td>0.12%</td>
<td>0.13%</td>
<td>0.13%</td>
</tr>
<tr>
<td>Annualised Total Risk</td>
<td>4.92%</td>
<td>4.19%</td>
<td>4.22%</td>
<td>4.90%</td>
<td>4.99%</td>
</tr>
<tr>
<td>Realised Sharpe Ratio</td>
<td>0.188</td>
<td>0.147%</td>
<td>0.334%</td>
<td>0.329%</td>
<td>0.315%</td>
</tr>
<tr>
<td>Annualised Tracking Error*</td>
<td>0.00%</td>
<td>1.53%</td>
<td>1.62%</td>
<td>0.86%</td>
<td>1.07%</td>
</tr>
<tr>
<td>Average Specific Risk Forecast</td>
<td>1.23%</td>
<td>0.84%</td>
<td>0.95%</td>
<td>1.32%</td>
<td>1.46%</td>
</tr>
<tr>
<td>Average Total Risk Forecast</td>
<td>4.41%</td>
<td>4.31%</td>
<td>4.34%</td>
<td>4.44%</td>
<td>4.49%</td>
</tr>
<tr>
<td>Portfolio Leverage</td>
<td>100.2%</td>
<td>107.5%</td>
<td>114.8%</td>
<td>109.8%</td>
<td>87.4%</td>
</tr>
<tr>
<td>Monthly One-Way Turnover</td>
<td>22.8%</td>
<td>24.7%</td>
<td>9.9%</td>
<td>9.9%</td>
<td>10.7%</td>
</tr>
</tbody>
</table>

*Note:* Relative to Full Replication.
Factor portfolio leverage (gross exposure to net asset value) ranged between 89.7% and 112.0% for the value portfolios and between 87.4% and 114.8% for the momentum portfolios. Thus, during the observed period, these factor-mimicking portfolios did not require high leverage in their implementation.\[^{10}\]

**FACTOR-MIMICKING PORTFOLIOS: APPLICATIONS**

In this section, we illustrate both passive and active management implementation of factor-mimicking portfolios.

**Passive Investment Strategies**

A simple way to capture factor returns would be to build a portfolio with stocks that have high exposure to a particular factor. For example, a value index such as the MSCI World Value Index, which only contains stocks that have been screened on different valuation ratios, could be used as a proxy to capture the value risk premium. Also, a small-cap index, such as the MSCI World Small Cap Index, could be used as a proxy to capture the size risk premium, but a simple approach based on screening typically leads to portfolios that, in addition to the target factor, also have significant exposures to other factors.

Exhibit 3 provides a snapshot of the factor exposures of the passive benchmarks. Indeed, a passive allocation to the MSCI World Value Index would result in small, but nonzero, exposures to other factors and high exposure to the financial sector. Similarly, a passive allocation to the MSCI World Small Cap Index would result in significant positive exposure to the variability in markets risk factor and somewhat smaller negative exposure to the value and momentum factors. Plan sponsors may wish to explicitly recognize these exposures when building their passive, optimal allocations to beta factors.

Factor-mimicking portfolios can be used as an overlay to benchmarks to isolate the risk premia investors aim to capture. In Exhibit 4, we provide a snapshot of the risk characteristics of the MSCI World Small Cap benchmark. We then provide the risk characteristics of this benchmark after sequentially hedging each of the unintended factor exposures using full replication. The final hedged portfolio retains some of the key risk characteristics of the

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**EXHIBIT 2**

Performance of Five Replicating Portfolios in Extreme Market Conditions

<table>
<thead>
<tr>
<th>Period</th>
<th>Event</th>
<th>Full Replication*</th>
<th>Total Risk Optimization**</th>
<th>Total Risk Optimization 10% Turnover**</th>
<th>Active Risk Optimization 10% Turnover**</th>
<th>Active Risk Optimization 400 Assets**</th>
</tr>
</thead>
<tbody>
<tr>
<td>GEM Value Factor Portfolios</td>
<td>Oct-97 Asian Crisis</td>
<td>1.46%</td>
<td>−0.17%</td>
<td>−0.19%</td>
<td>−0.08%</td>
<td>−0.18%</td>
</tr>
<tr>
<td></td>
<td>Sep-98 LTCM</td>
<td>−1.07%</td>
<td>0.56%</td>
<td>0.54%</td>
<td>−0.06%</td>
<td>−0.22%</td>
</tr>
<tr>
<td></td>
<td>Apr-00 TMT Bubble</td>
<td>1.39%</td>
<td>−0.45%</td>
<td>−0.46%</td>
<td>−0.06%</td>
<td>−0.33%</td>
</tr>
<tr>
<td></td>
<td>Oct-01 WTC Attack</td>
<td>−0.42%</td>
<td>−0.52%</td>
<td>−0.86%</td>
<td>−0.36%</td>
<td>−0.29%</td>
</tr>
<tr>
<td></td>
<td>Nov-02 Enron</td>
<td>0.89%</td>
<td>0.17%</td>
<td>0.15%</td>
<td>−0.02%</td>
<td>0.23%</td>
</tr>
<tr>
<td></td>
<td>Apr-03 Iraq War</td>
<td>0.85%</td>
<td>−0.18%</td>
<td>−0.11%</td>
<td>0.02%</td>
<td>0.14%</td>
</tr>
<tr>
<td></td>
<td>Sep-07 Quant &quot;Meltdown&quot;</td>
<td>−1.31%</td>
<td>0.37%</td>
<td>0.37%</td>
<td>0.00%</td>
<td>0.16%</td>
</tr>
<tr>
<td>GEM Momentum Factor Portfolios</td>
<td>Sep-97 Asian Crisis</td>
<td>1.74%</td>
<td>0.14%</td>
<td>0.24%</td>
<td>0.09%</td>
<td>0.08%</td>
</tr>
<tr>
<td></td>
<td>Oct-98 LTCM</td>
<td>−3.59%</td>
<td>0.67%</td>
<td>1.20%</td>
<td>0.42%</td>
<td>0.19%</td>
</tr>
<tr>
<td></td>
<td>Mar-00 TMT Bubble</td>
<td>−4.39%</td>
<td>1.84%</td>
<td>1.72%</td>
<td>0.14%</td>
<td>0.58%</td>
</tr>
<tr>
<td></td>
<td>Oct-01 WTC Attack</td>
<td>−0.71%</td>
<td>0.59%</td>
<td>0.60%</td>
<td>0.66%</td>
<td>0.80%</td>
</tr>
<tr>
<td></td>
<td>Nov-02 Enron</td>
<td>−4.15%</td>
<td>1.00%</td>
<td>1.11%</td>
<td>0.25%</td>
<td>0.28%</td>
</tr>
<tr>
<td></td>
<td>Apr-03 Iraq War</td>
<td>−3.41%</td>
<td>0.27%</td>
<td>0.38%</td>
<td>0.26%</td>
<td>0.14%</td>
</tr>
<tr>
<td></td>
<td>Sep-07 Quant &quot;Meltdown&quot;</td>
<td>1.30%</td>
<td>0.05%</td>
<td>0.26%</td>
<td>0.12%</td>
<td>0.21%</td>
</tr>
</tbody>
</table>

Notes: *Absolute return; **Return relative to Full Replication.
### Exhibit 3
Factor Exposures of Passive Benchmarks, Year-End 2007

<table>
<thead>
<tr>
<th>MSCI World Style Indices</th>
<th>Barra GEM Factor Portfolios</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Value</td>
</tr>
<tr>
<td><strong>GEM Factor Exposures</strong></td>
<td></td>
</tr>
<tr>
<td>(Standard Deviation)</td>
<td></td>
</tr>
<tr>
<td>Size Factor</td>
<td>0.24</td>
</tr>
<tr>
<td>Momentum Factor</td>
<td>-0.27</td>
</tr>
<tr>
<td>Value Factor</td>
<td>0.56</td>
</tr>
<tr>
<td>Volatility Factor</td>
<td>-0.29</td>
</tr>
</tbody>
</table>

**GICS Sector Exposures** (%)
- Consumer Discretionary: -1.53
- Consumer Staples: -2.63
- Energy: 4.09
- Financials: 14.30
- Health Care: -1.72
- Industrials: -3.11
- Information Technology: -9.67
- Materials: -3.57
- Telecom Services: 2.45
- Utilities: 1.38

Notes: *Factor exposures greater than 0.2 standard deviation are typically considered significant exposures; **Sector exposure is relative to MSCI World for the style indices and relative to cash for the Barra factor portfolios.

### Exhibit 4
Barra GEM Ex Ante Risk Analysis, Year-End 2007

<table>
<thead>
<tr>
<th>MSCI World Small Cap</th>
<th>Momentum Factor Portfolio</th>
<th>Value Factor Portfolio</th>
<th>Volatility Factor Portfolio</th>
<th>MSCI SC + Value Hedge</th>
<th>MSCI SC + Mom Hedge</th>
<th>MSCI SC + Vola Hedge</th>
</tr>
</thead>
<tbody>
<tr>
<td>Risk Factor Exposures</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Size Factor</td>
<td>-3.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>-3.00</td>
<td>-3.00</td>
</tr>
<tr>
<td>Momentum Factor</td>
<td>-0.17</td>
<td>1.00</td>
<td>0.00</td>
<td>0.00</td>
<td>-0.17</td>
<td>0.00</td>
</tr>
<tr>
<td>Value Factor</td>
<td>-0.22</td>
<td>0.00</td>
<td>1.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Volatility Factor</td>
<td>1.19</td>
<td>0.00</td>
<td>1.00</td>
<td>1.19</td>
<td>1.19</td>
<td>0.00</td>
</tr>
</tbody>
</table>

**Forecast Risk Decomposition**
- Risk Indices Total: 7.19%
- Size Factor: 7.11%
- Momentum Factor: 0.64%
- Value Factor: 0.49%
- Volatility Factor: 3.28%
- Industry Risk: 1.27%
- Country Risk: 14.49%
- Currency Risk: 2.95%
- Specific Risk: 0.64%
- Total Risk: 17.60%
benchmark—identical country, industry, and currency risk—and similar total risk; however, by mitigating the value, success, and variability in markets style factors, we obtain a portfolio that targets the global size premium.

Exhibits 3 and 4 provide a snapshot of risk at a point in time. Thus, we wondered what would happen if we hedged the unintended style exposures to the MSCI World Small Cap Index over time. Because this type of analysis is directed at passive investing, we utilized full replication on a monthly basis for the period December 2003 to December 2008. The results were somewhat startling. Over the period of analysis, the MSCI World Small Cap Index returned 2.54% a year with a volatility of 17.9%, whereas the hedged MSCI World Small Cap Index returned 6.25% a year with a volatility of 17.6%. The outperformance of the hedged benchmark is largely attributable to eliminating exposure to the volatility risk factor, which had negative return. Though this outperformance cannot be guaranteed into the future, our experiment illustrates the importance of identifying and isolating the relevant beta premium when constructing optimal, passive allocations.

Active Investment Strategies

Fundamental active investment strategies are often characterized as being top down or bottom up. Bottom-up strategies emphasize security selection. Analysts will rate the relative attractiveness of individual companies on the basis of balance sheet fundamentals, expected future cash flows, and the quality of management in the context of the current environment. Portfolio managers of bottom-up investment processes build their active exposures by going long the top analyst recommendations and going short the “dogs” (stocks that are expected to underperform), while trying to maintain balanced or minimal exposures to sectors and investment styles.

Factor-mimicking portfolios can be used to mitigate style exposures allowing portfolio managers to capture pure alpha. To illustrate this concept, we consider a hypothetical strategy in which the investment manager has perfect foresight over a 12-month forecast horizon for U.S. stocks. In order to exploit these views, the manager constructs an absolute return investment strategy, going long the 50 expected best-performing stocks and shorting the 50 expected worst performers, on an equal-weighted basis. These portfolios should deliver large positive returns (by construction), but they may be exposed to large systematic risks.

In Exhibit 5, we provide summary statistics on the style exposures of these portfolios for the period December 1997 to June 2008, using the Barra USE3 model. As shown in the exhibit, the resulting portfolios have nonzero exposure to systematic risk factors. For instance, the long–short

<table>
<thead>
<tr>
<th>Style Exposures of Factor-Mimicking Portfolios</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>USE3 Risk Factors</strong></td>
</tr>
<tr>
<td>Factor Exposures of Long–Short Perfect-Foresight Portfolios</td>
</tr>
<tr>
<td>Average Exposure</td>
</tr>
<tr>
<td>Positive Exposure</td>
</tr>
<tr>
<td>Number of Months with Positive Exposure</td>
</tr>
<tr>
<td>Average</td>
</tr>
<tr>
<td>Negative Exposure</td>
</tr>
<tr>
<td>Number of Months with Negative Exposure</td>
</tr>
</tbody>
</table>

Factor Risk and Return

Realized Factor Return (% annualized) | 1.60 | 0.55 | −0.85 | 4.10 | −0.21 | −0.70 | −1.28 |
Average Forecast Risk (% annualized) | 6.97 | 5.16 | 2.56 | 3.02 | 1.52 | 1.68 | 1.49 |
strategy maintains an average short exposure to the volatility factor in 100 of the 126 months of the analysis. This exposure would have detracted from risk-adjusted performance because this factor earned about 160 bps a year and contributed to systematic portfolio risk; the average factor volatility was about 7% over this period.

The portfolio manager in our analysis might ask whether the returns from his strategy originate from stock-specific alpha—reflecting the stock selection skill of his analysts—or from alpha derived from exposure to systematic risk. Alternatively, the manager might ask whether the risk-adjusted performance of the strategy would be enhanced by hedging out the unwanted, or incidental, factor risk. In Exhibit 5, we noted that portfolios had significant exposure to volatility, momentum, and trading activity. These three factors were also among the most volatile factors during this period. Unintended exposure to these factors may reduce the risk-adjusted performance of the underlying stock-picking strategy by eroding the alpha and increasing the volatility of the portfolio.

We thus consider the following experiment. At the beginning of each month, the manager will hedge away the unintended systematic factor exposures to volatility, momentum, and trading activity by using factor overlay portfolios that neutralize the portfolio exposure to these factors. Clearly, it would be possible to hedge each of the systematic exposures, an approach that will be reviewed in future research. Our results, however, will suffice to illustrate the potential benefits of hedging common factor risk. Our hypothetical implementation is carried out with tracking portfolios that hold 400 assets and that have a maximum monthly turnover of 10%, as these characteristics best reflect how the factor-mimicking portfolios might be implemented in practice.

Exhibit 6 shows that the hedged perfect-foresight portfolios significantly outperform the unhedged portfolios by about 40% in terms of percentage gain in risk-adjusted performance. While the return of the hedged portfolios is slightly decreased, the decrease is more than compensated with a substantial reduction in total risk.

---

**Exhibit 6**

Perfect-Foresight Portfolio Performance, January 1997–June 2008

<table>
<thead>
<tr>
<th></th>
<th>Unheded Perfect Foresight Portfolio</th>
<th>Full Replication Overlay</th>
<th>Total Risk Optimization 10% Turnover Overlay</th>
<th>Active Risk Optimization 10% Turnover Overlay</th>
<th>Active Risk Optimization 10% Turnover 400 Assets Overlay</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Long–Short Perfect-Foresight Portfolio Performance</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average Return to</td>
<td>0.337</td>
<td>0.519</td>
<td>0.490</td>
<td>0.493</td>
<td>0.481</td>
</tr>
<tr>
<td>% Gain in Return to</td>
<td>-</td>
<td>53.8</td>
<td>45.3</td>
<td>46.2</td>
<td>42.5</td>
</tr>
<tr>
<td>Average Return to</td>
<td>0.526</td>
<td>0.730</td>
<td>0.705</td>
<td>0.705</td>
<td>0.677</td>
</tr>
<tr>
<td>% Gain in Return to</td>
<td>-</td>
<td>39.0</td>
<td>34.2</td>
<td>34.1</td>
<td>28.8</td>
</tr>
<tr>
<td>Realized Return (Monthly, %)</td>
<td>9.34</td>
<td>8.79</td>
<td>8.59</td>
<td>8.61</td>
<td>8.49</td>
</tr>
<tr>
<td>Realized Risk (Annualized, %)</td>
<td>27.69</td>
<td>16.94</td>
<td>17.52</td>
<td>17.46</td>
<td>17.67</td>
</tr>
<tr>
<td>Average Forecast Risk (Annualized, %)</td>
<td>17.77</td>
<td>12.03</td>
<td>12.18</td>
<td>12.22</td>
<td>12.55</td>
</tr>
<tr>
<td>Average Common Factor Risk (Annualized, %)</td>
<td>16.08</td>
<td>9.60</td>
<td>9.60</td>
<td>9.60</td>
<td>9.63</td>
</tr>
<tr>
<td>Average Specific Risk (Annualized, %)</td>
<td>7.02</td>
<td>7.07</td>
<td>7.34</td>
<td>7.40</td>
<td>7.90</td>
</tr>
</tbody>
</table>
from 18% to 12%, mainly coming from reduced common factor risk; specific risk is only slightly increased, if changed at all. Moreover, when monitoring performance at the end of each calendar year, Exhibit 7 shows that the hedged portfolios outperform the perfect-foresight strategy every year during the backtest. Interestingly, the gains in yearly cumulative returns to risk seem to be higher in periods of increased market volatility.

Indeed, when market volatility peaked in 2002, a significant portion of the risk of the unhedged portfolio came from volatility and momentum, with average contributions to total risk from these two factors being 58% and 35%, respectively. Hedging away exposure to these two factors significantly improved the portfolio’s risk-adjusted performance in that year. In fact, the substantial improvement in risk-adjusted performance from hedging common factor risk in 2002 can be largely attributed to the months of October and, especially, November.

Exhibit 8 plots unhedged versus hedged portfolio returns over the entire backtesting period. This exhibit shows that despite having perfect foresight over 12-month horizons, the manager of the unhedged long–short portfolio would still have experienced 11 different months over the backtest period of negative monthly returns as well as a maximum drawdown of approximately –35% during the period October–November 2002. Even our “super manager” may not survive losing a third of his assets over a two-month period! But the “prudent manager” who used factor overlays to hedge her portfolio’s exposure to the most volatile sources of common factor risk would have experienced only a –5% drawdown over the same two-month period. The option-like payoff structure of the hedged portfolio illustrated in this diagram is clearly of great interest.

**IMPLICATIONS FOR PORTFOLIO MANAGEMENT**

We have presented alternative methods for constructing factor-replicating portfolios. We addressed the issue of implementation cost by recognizing that portfolio turnover (and therefore transaction costs) is an important parameter to control in the construction of constrained factor portfolios. Importantly, we demonstrated that constrained factor portfolios with a limited number of assets and relatively low turnover track pure factor returns reasonably well and, therefore, can serve as an investment instrument for factor-based hedging or to obtain beta exposure to a particular factor.

We have illustrated how factor-mimicking portfolios can be applied to both passive and active investment strategies. Factor-mimicking portfolios can be utilized to hedge out the unintended factor exposures
of conventional benchmarks aimed at targeting a particular beta factor. This ability enables plan sponsors to better manage their optimal allocations to beta factor risks. Factor-mimicking portfolios can also be utilized to hedge out the style exposures of active stock-picking strategies, which enables the active manager to capture pure alpha. In brief, the practice of both active and passive investment management can be enhanced via factor-mimicking portfolios.

**APPENDIX**

**Equivalence between Specific Risk Optimization and Total Risk Optimization**

In this appendix, we show that specific risk–weighted and total risk–weighted factor portfolios are the same up to a scaling parameter. We also show how factor portfolio weights can be computed either through generalized least squares (GLS) regression or constrained optimization.

**Defining performance criteria.** In order to assess the performance of optimized replication portfolios, we need to define appropriate performance criteria that reflect the objectives of the factor replication problem, namely, maximum exposure to the targeted factor and minimum portfolio risk. This reasoning leads us, similar to Sharpe [1975], to consider two reward-to-variability performance criteria that reflect these objectives. The two performance criteria we consider are the ratio of portfolio exposure to the targeted factor divided by specific risk ($J_S$) and the ratio of portfolio exposure to the targeted factor divided by total risk ($J_T$).

$$J_S(h) = \frac{h'X_a}{(h'Dh)^{1/2}}$$

$$J_T(h) = \frac{h'X_a}{(h'Vh)^{1/2}}$$

**Specific risk approach.** First, we consider the specific risk performance criterion $J_S$. The standard approach for constructing portfolios that maximize this criterion and have zero exposure to the remaining risk factors involves constrained long–short optimization. In this optimization, asset exposures to the targeted factor play the role of expected returns while constraints are imposed to control portfolio exposure to the remaining risk factors. Using our general framework, the weights that maximize criterion $J_S$ are given by the following expression:
\[ h_i^* = \frac{1}{\lambda} D^{-1} \alpha = \frac{1}{\lambda} D^{-1} [X_a - X_a (X_a' D^{-1} X_a)^{-1} (X_a' D^{-1} X_a)] \]

**Total risk approach.** Next, we consider the problem of maximizing the total risk performance criterion, \( J_t \), subject to having zero exposure to all other factors. As in the specific risk case, this problem can be solved through constrained optimization that maximizes portfolio exposure to the targeted factor for a given level of total risk, subject to constraints on all remaining risk factors. The weights that maximize criterion \( J_t \) are as follows:

\[ h_i^* = \frac{1}{\lambda} V^{-1} \alpha = \frac{1}{\lambda} V^{-1} [X_a - X_a (X_a' V^{-1} X_a)^{-1} (X_a' V^{-1} X_a)] \]  \hspace{1cm} (A-3)

**Reconciling the specific risk and total risk problems.** By construction, the common factor risk, \( \sigma_{CF} \), of factor-replicating portfolios depends only on the exposure, \( x_a \), of these portfolios to the targeted factor and the risk, \( \sigma_{CF} \), of the targeted factor. In other words, factor-replicating portfolios are not exposed to common factor risk due to the variance and covariance of other factors except the targeted factor. As a result, we can express the common factor risk, \( \sigma_{CF} \), of factor-replicating portfolios as follows:

\[ \sigma_{CF} = \lambda x_a \sigma_{CF} \]  \hspace{1cm} (A-4)

This expression enables us to rewrite the total risk performance criterion, \( J_t \), as follows:

\[ J_t(h) = \frac{h'X_a}{(h'h)^{1/2}} = \frac{X_a}{\left( \sigma_{CF}^2 + h'Dh \right)^{1/2}} \]

\[ = \frac{1}{\left( \frac{\lambda \sigma_{CF}^2}{\sigma_{CF}^2} + \frac{(\lambda^2 - 1)/2}{\sigma_{CF}^2} \right)^{1/2}} \left( \sigma_{CF}^2 + h'Dh \right)^{1/2} \]  \hspace{1cm} (A-5)

The last equation demonstrates that performance criteria, \( J_s \) and \( J_t \), constrained to \( h'X_a = 0 \), are maximized by the same optimal portfolio up to a scaling factor; in other words,

\[ h_i^* = \gamma h_i^s \]  \hspace{1cm} (A-6)

**Equivalence between constrained optimization and GLS regression.** Factor returns estimated through the GLS regression expressed in Equation (1) with weights \( V^{-1} \) are given by

\[ \hat{f} = (X' V^{-1} X)^{-1} X' V^{-1} r \]  \hspace{1cm} (A-7)

The component of factor returns, \( f_{\alpha} \), corresponding to the target factor can be estimated through a two-step regression process. In the first step, we regress target factor exposures, \( X_a \), on all other factor exposures, \( X_a \), using the same weighting matrix \( W^{-1} \),

\[ X_a = X_a \hat{b} + \alpha \]  \hspace{1cm} (A-8)

The residuals from this weighted least squares regression can be estimated as follows:

\[ \hat{\alpha} = X_a - X_a \hat{b} = X_a - X_a (X_a' W^{-1} X_a)^{-1} (X_a' W^{-1} X_a) \]  \hspace{1cm} (A-9)

In the second step, we regress asset returns on the residuals from the first-step regression,

\[ r = \hat{\alpha} f_a + e \]  \hspace{1cm} (A-10)

The estimated target factor return component can then be written as follows:

\[ \hat{f}_{\alpha} = (\hat{\alpha} W^{-1} \hat{\alpha})^{-1} \hat{\alpha} W^{-1} r = h_i^t r \]  \hspace{1cm} (A-11)

We can see that the factor-mimicking portfolio weights, \( h_i^t \), are also the solution to the constrained optimization problem expressed in Equations (4) and (5) with a scaling parameter, \( \lambda \), equal to \( \alpha' W^{-1} \alpha \),

\[ h_i = \frac{1}{\lambda} W^{-1} [X_a - X_a (X_a' W^{-1} X_a)^{-1} (X_a' W^{-1} X_a)] \]

\[ = (\alpha' W^{-1} \alpha)^{-1} W^{-1} \alpha \]  \hspace{1cm} (A-12)

**ENDNOTES**

This article was completed while Stefano Cavaglia was employed as the head of Quantitative Strategies at UBS O’Connor LLC.

1In Equation (1), \( r \) is the \( n \times 1 \) vector of asset excess returns, \( X \) is the \( n \times k \) matrix containing asset exposures to fundamental factors, \( f \) is the \( k \times 1 \) vector of factor returns, \( \epsilon \) is the \( n \times 1 \) vector of specific returns, \( n \) is the number of assets in the estimation universe, and \( k \) is the number of factors in the model. In Equation (2), \( F \) is the \( k \times k \) factor return covariance matrix, \( D \) is the \( n \times n \) specific return covariance matrix, and \( V \) is the \( n \times n \) asset return covariance matrix. Asset exposures to fundamental factors are typically represented by dummy variables for industry factors and cross-sectional z-scores for style factors; for example, size, value, growth, momentum, and so on. The specific return covariance matrix \( D \) is usually assumed to be diagonal. However, some off-diagonal elements may be
different from zero, for example, elements corresponding to securities issued by the same company. In our analysis we do not need to impose any limiting assumptions on the structure of the specific return covariance matrix.

3From a theoretical perspective, regression coefficients estimated through unweighted least squares regression are BLUE (best linear unbiased estimators) if the errors are uncorrelated with each other and with the independent variables, and they have equal variance; if the errors have different variance, however, then weighted least squares regression using the inverse of the error variances as weights leads to regression coefficients that are BLUE.

4For further details on the full replication approach, see Grinold and Kahn [1995, p. 74].

5Institutional constraints limit portfolio leverage for certain regulated mutual funds to 2:1. For detailed analysis of the impact of leverage constraints on portfolio efficiency, see Melas and Suryanarayanan [2008].

6For example, the 10/40 rule under the UCITS regulation in Europe restricts the amount of capital a fund can allocate to a single asset to less than 10% and restricts the total amount allocated to assets above 5% to less than 40% of the net asset value of the portfolio.

7In addition to standard transaction costs (commission and market impact) that are a linear or power function of traded volume, other transaction costs, for example, ticket costs and custody fees, are a function of the number of securities traded or held in the portfolio.

8In the numerical simulations, we used MATLAB and the Barra Open Optimizer.

9The statistics reported in Exhibit 2 highlight the performance of different replication methods over monthly investment horizons. In certain hedging applications, investors may also wish to ensure that the hedging portfolios track well the underlying factor returns over shorter horizons. In order to assess tracking error over daily investment horizons, we use data from the Barra U.S. Equity Model (USE3) to assess the daily performance of different factor-replicating methods during the market turmoil of August 2007. The analysis, which is available on request, suggests that our broad conclusions remain unchanged at the daily frequency.

10It may seem surprising at first that, in all cases, leverage fluctuates around 100%. This is due to the fact that factor portfolios maintain unit exposure to the target factor. In order to analytically illustrate this point, we assume that all assets have the same specific risk. Then, optimal weights of otherwise unconstrained optimized replication portfolios may be expressed as follows:

\[ h_i^* = \frac{1}{\delta} \alpha_i \lambda \]  

(E-1)

Here, \( \delta \) is the specific risk of asset \( i \), and \( \alpha_i \) is the residual from the regression of target factor exposures on exposures to all the other factors. So, the scaling factor \( \lambda \) is given by

\[ \sum_{i=1}^{n} h_i^* \alpha_i = 1 \Rightarrow \lambda = \sum_{i=1}^{n} \frac{\alpha_i}{\sigma_i^2} \]  

(E-2)

Then, for large number of assets \( n \), we can compute factor portfolio leverage as follows:

\[ L = \sum_{i=1}^{n} |h_i^*| \approx \frac{\sum_{i=1}^{n} |\alpha_i|}{\sum_{i=1}^{n} \frac{\alpha_i^2}{\sigma_i^2}} = \frac{\sigma_\alpha^2}{\sigma^2} = 2 \frac{1}{\sigma^2} \frac{1}{\pi} \]  

(E-3)

Here, we assumed that the residuals are approximately normally distributed across assets, with zero mean and \( \sigma \) standard deviation. Empirically, we find that \( \sigma_\alpha \) is equal to 0.78, on average, for the value and momentum factors, which leads to a leverage of 100%.

REFERENCES


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