Maximizing the Sharpe Ratio and Information Ratio in the Barra Optimizer

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INTRODUCTION

Optimization techniques can provide significant assistance to an investor in the portfolio construction process. In particular, they help accommodate each investor's specific objective. According to the canonical Markowitz portfolio optimization model, investors should choose among portfolios that belong to the efficient frontier. This means that a portfolio should have the minimum risk for a given level of return, or the maximum return for a given level of risk. In practice, investors will have to deal with the trade-off between portfolio risk and return.

The traditional mean-variance portfolio optimization methodology offers a uniform way of dealing with this trade-off. Its goal is to find a portfolio that satisfies all constraints while maximizing the following risk-adjusted return function (i.e., utility function):

Utility function = portfolio return - risk aversion * portfolio variance,

where risk aversion is a parameter reflecting the investor's risk tolerance or preference.

An alternative approach is to choose a portfolio that, instead of maximizing the risk-adjusted return, maximizes the ratio of return to risk, or the ratio of active return to the tracking error. In other words, the objective is to find a portfolio that maximizes either the Sharpe Ratio or Information Ratio.

The Barra Optimizer can assist investors in the process of either portfolio construction or rebalancing. It can accommodate both mean-variance and ratio objectives: users can minimize tracking error, maximize risk-adjusted return, or maximize the Sharpe Ratio or Information Ratio.

The rest of this article is structured as follows: first, we will formally describe the problems the Barra Optimizer will solve when users want to maximize mean-variance and ratio objectives. Then we will compare and examine the solutions obtained by optimizing various objective functions in a simple but typical portfolio construction problem. We will also explain the algorithms for portfolio optimization that achieve Sharpe Ratio and Information Ratio maximization.

OBJECTIVE FUNCTIONS IN DETAIL

Barra Optimizer users can choose to maximize either a standard mean-variance utility function or the *Information Ratio* or *Sharpe Ratio* of their portfolios. The portfolio optimization problems with these objective functions have the following general form, described below.

Maximizing the Risk-Adjusted Return

The objective in the standard mean-variance portfolio optimization problem is to maximize risk-adjusted return. In the active setting (i.e., in the presence of a benchmark) the problem can be formulated as follows:

$$
(\mathrm{MAV}_\lambda) \quad \begin{array}{ll} \text{Maximize} & r^T(h-h_B)-\lambda(h-h_B)^T V(h-h_B)-TC(h) \\ \text{s.t.} & l_a \le Ah \le u_a, \\ & l_h \le h \le u_h, \end{array}
$$

where *h* is a holding vector, r is a return vector, h_B is benchmark, λ is a risk aversion, *V* is covariance matrix, l_a, l_h, u_a, u_h are bounds; *A* is the coefficient matrix for linear constraints (including factor constraints and the budget constraint) and $TC(h)$ is the transaction cost term. Note that the active return term $r^T(h-h_B)$ in the objective can be replaced by $r^T h$, since $r^T h_B$ is constant and will not affect the results of optimization.

When the benchmark is not present, the problem becomes:

$$
(\mathrm{MV}_\lambda) \qquad \begin{array}{ll} \text{Maximize} & r^T h - \lambda h^T V h - TC(h) \\ \text{s.t.} & l_a \le Ah \le u_a, \\ & l_h \le h \le u_h. \end{array}
$$

Maximizing the Sharpe Ratio

The Sharpe Ratio problem -- (SR) for short -- is defined as the excess return of a portfolio divided by its total risk. The portfolio optimization problem that maximizes the Sharpe Ratio has the following form:

$$
(\mathrm{SR}) \qquad \begin{array}{ll} \text{Maximize} & \dfrac{r^{\mathrm{T}} h - r_f - TC(h)}{\sqrt{h^{\mathrm{T}} V h}} \\ \text{s.t.} & l_a \le Ah \le u_a, \\ & l_h \le h \le u_h, \end{array}
$$

where r_f is the risk-free interest rate.

Maximizing the Information Ratio[1]

The Information Ratio problem -- (IR) for short -- is defined as the active return of a portfolio divided by its tracking error. The portfolio optimization problem that maximizes the Information Ratio has the following form:

$$
(\mathrm{IR}) \qquad \begin{array}{ll} \text{Maximize} & \dfrac{r^{\mathrm{T}}(h-h_B) - TC(h)}{\sqrt{(h-h_B)^{\mathrm{T}} V (h-h_B)}} \\ \text{s.t.} & l_a \le Ah \le u_a, \\ & l_h \le h \le u_h. \end{array}
$$

OBJECTIVE FUNCTION COMPARISON

To construct a portfolio that suits their needs, users can maximize either the standard mean-variance utility function or the ratio type function. In this section, we will demonstrate the

[1] The IR definition used here is different from the one given by Grinold and Kahn, who denote by IR the ratio of portfolio residual return to portfolio residual risk.

relationship between these two types of portfolio optimization problems. We will first discuss the relationship conceptually, then use a two-stock case for illustration.

Because optimization problems maximizing the Sharpe Ratio and Information Ratio have similar properties, we will hereafter concentrate on discussing the problem that maximizes the Sharpe Ratio. The following discussion can be applied to a problem that maximizes the Information Ratio, except when mentioned otherwise.

Assume that h^* is the optimal portfolio maximizing the Sharpe Ratio problem.

Let $\hat{\lambda} = \dfrac{r^T h^* - r_f - TC(h^*)}{2 * h^{*T} V h^*}$.

From **Theorem 1** in the **Appendix**, we know that h^* is also an optimal portfolio for the standard mean-variance portfolio optimization problem -- $(\text{MV}_{\hat{\lambda}})$ for short. Therefore, the following holds:

There exists a risk aversion $\hat{\lambda}$ such that portfolio optimization problems for (SR) and $(\text{MV}_{\hat{\lambda}})$ have the same optimal portfolio.

In general, we have the following relationship between portfolio optimization problems (MV_{λ}) and (SR):

(1): As intended, solving the Sharpe Ratio maximization problem (SR) will produce a portfolio with the maximum Sharpe Ratio. Ordinarily, the optimal portfolio in the standard mean-variance portfolio optimization problem (MV_{λ}) is not necessarily the one with the maximum Sharpe Ratio. If we use the correct risk aversion, the optimal portfolio of (MV_{λ}) will have the maximum Sharpe Ratio, too.

(2): The standard mean-variance portfolio optimization problem (MV_{λ}) is more flexible. By setting proper risk aversion, users will get the portfolio that suits their risk tolerance and maximizes the risk-adjusted return.

For a “plain vanilla” mean-variance portfolio optimization problem, follow this:

$$\begin{aligned} \text{Maximize} \quad & r^T h - \lambda h^T V h \\ \text{s.t.} \quad & e^T h = 1, \end{aligned}$$

where it is possible to obtain a closed-form formula for the optimal portfolio:

$$h(\lambda) = \frac{V^{-1} r}{2\lambda} + \left(1 - \frac{e^T V^{-1} r}{2\lambda}\right) \frac{V^{-1} e}{e^T V^{-1} e}.$$

As the risk aversion increases, the optimal portfolio $h(\lambda)$ increasingly resembles the minimum risk portfolio, i.e., $h_C = V^{-1} e / e^T V^{-1} e$.

As the risk aversion decreases, users will take more risk. At the extreme, when the risk aversion is zero, one completely disregards the risk. The optimal portfolio in a long-only case will hold only the stock that has the largest alpha. Figure 1 illustrates the relative positions of minimum risk, maximum Sharpe Ratio, and maximum return portfolios on the risk-return frontier.

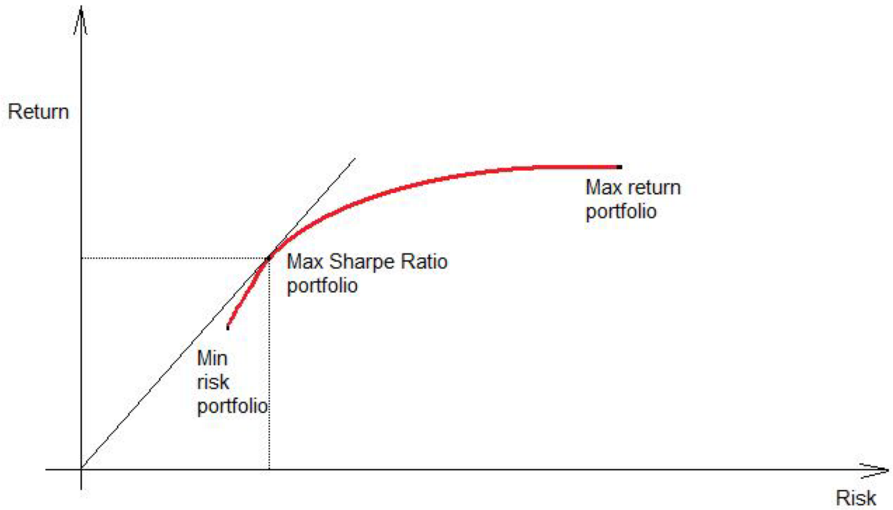

Figure 1. Maximum Sharpe Ratio

Case Study

Now let's look at a two-stock portfolio optimization problem and compare the optimal solutions of the two problem types. For simplicity, let's limit the discussion to the long-only case and assume there are no constraints besides the total investment constraint. We also ignore the transaction costs and assume the risk-free rate is 0.

The optimization problems to solve in this case study are:

$$\text{(MV)} \quad \text{Maximize} \quad r^{\mathrm{T}}h - \lambda h^T V h$$
$$\text{s.t.} \quad e^T h = 1,\ h \geq 0.$$
$$\text{(SR)} \quad \text{Maximize} \quad \frac{r^{\mathrm{T}}h}{\sqrt{h^T V h}}$$
$$\text{s.t.} \quad e^T h = 1.0,\ h \geq 0,$$

where:

$$r = \begin{pmatrix} 0.05 \\ 0.1 \end{pmatrix},\ V = \begin{pmatrix} 0.05 & 0.02 \\ 0.02 & 0.07 \end{pmatrix},\ e = \begin{pmatrix} 1.0 \\ 1.0 \end{pmatrix}.$$

The optimal portfolio for the Sharpe Ratio problem (SR) is:

$$h_{SR} = \begin{pmatrix} 0.2727 \\ 0.7273 \end{pmatrix},\ r^T h_{SR} = 0.08636,\ \sigma_{SR} = 0.2206,\ \frac{r^T h_{SR}}{\sigma_{SR}} = 0.3915.$$

The following table presents optimal solutions and other pertinent information about the (MV) problem with different levels of risk aversion.

	Portfolio 1	Portfolio 2	Portfolio 3	Portfolio 4	Portfolio 5	Portfolio 6	Portfolio 7	Portfolio 8
λ Risk Aversion (input)	0.1	0.6	0.8	**0.887** **($\hat{\lambda}$)**	1.0	2.0	7.5	50
h_1 Asset 1 Weight (output)	0.0	0.104	0.234	**0.2727**	0.313	0.469	0.583	0.619
h_2 Asset 2 Weight (output)	1.0	0.896	0.766	**0.7273**	0.687	0.531	0.417	0.381
$r^T h$ (%) Portfolio Return	10.0	9.479	8,828	**8.636**	8.437	7.656	7.083	6.906
$\sqrt{h^T V h}$ (%) Portfolio Risk	26.46	24.59	22.574	**22.06**	21.58	20.175	19.72	19.686
$\frac{r^T h}{\sqrt{h^T V h}}$ Sharpe Ratio	0.378	0.3855	0.3911	**0.3915**	0.391	0.3795	0.3592	0.3508

In the first row labeled Risk Aversion, we have applied varying levels of λ, creating eight (8) different portfolios. The subsequent rows display corresponding results for holding weights (Assets 1 and 2), portfolio return, risk, and the Sharpe Ratio.

This table illustrates that when risk aversion is 0.1, the optimal portfolio will hold 100% of stock 2 and the portfolio return is 10% -- the largest possible portfolio return. As the risk aversion increases, the optimal portfolio will increasingly hold more of stock 1, while both portfolio return and risk will decrease. However, the Sharpe Ratio will increase at first, then steadily decrease. As illustrated above, the maximum Sharpe Ratio is reached when risk aversion is **0.887 (see under Portfolio 4)**. When risk aversion is equal to or larger than 50, an optimal portfolio will be same as a minimum risk portfolio.

As designed, solving a Sharpe Ratio (SR) problem will provide a portfolio with the maximum Sharpe Ratio. Solving a standard mean-variance optimization problem (MV_λ) does not necessary get the maximum Sharpe Ratio portfolio. However, (MV_λ) will give users more control on the risk level of the optimal portfolio.

APPROACHES AND ALGORITHMS

There are several algorithms for solving the maximum Sharpe Ratio or Information Ratio problem. For example, the nonlinear programming approach (direct solution), the line search approach (including Newton's method and Golden Section search), and the homogenization approach.

4.1. The Nonlinear Programming Approach (Direct Solution)

This approach utilizes a nonlinear programming solver to optimize the Sharpe Ratio or Information Ratio directly. Because the nonlinear terms appear only in the objective function and all constraints are linear or piecewise-linear, we can use, for example, a **reduced-gradient algorithm** in conjunction with a **quasi-Newton algorithm** for solving this kind of problem. The nonlinear programming approach has, essentially, no limitations. However, in cases where the objective can never assume positive values, the problem might become non-convex, which removes an optimality guarantee and might result in numerical difficulties.

4.2. Line Search Approach

This approach solves a sequence of standard portfolio optimization (MV_{λ}) problems. As illustrated by **Theorem 1** in the **Appendix,** under certain conditions we get the optimal portfolio for a Sharpe Ratio problem (SR) by solving a standard portfolio optimization problem with the correct risk aversion. It is also illustrated by **Theorem 2** in the **Appendix** that the Sharpe Ratio for portfolios on the efficient frontier is first increasing the risk aversion, until it reaches the maximum, and then it starts decreasing. Therefore, the Golden Section search algorithm can be utilized to find the optimal risk aversion.

4.3. Homogenization Approach

A Sharpe Ratio maximization problem (SR) without transaction costs and penalties can be solved quickly (solving one linear programming problem and one convex quadratic programming problem) by utilizing the homogenization method described by **Theorem 3** in the **Appendix**. However, this method has certain drawbacks: piecewise linear transaction costs, presence of a benchmark, and even asset-variable bounds that need special handling. For instance, accommodating piecewise linear transaction costs will require splitting asset-holding variables, which will result in increased problem size.

SUMMARY

Besides the standard mean-variance portfolio optimization that maximizes the risk-adjusted return, the Barra Optimizer offers users an option to maximize the Sharpe Ratio or Information Ratio. However, the standard mean-variance portfolio optimization is more flexible. It gives users a control on the risk level of the optimal portfolio. Also, users will be able to use more functions and features in the Barra Optimizer when maximizing risk-adjusted return.

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APPENDIX

Theorem 1: Assume that we want to solve a problem:

$$\max \frac{A(x)}{\sqrt{B(x)}} \qquad (1)$$
$$g_i(x) \le 0, \qquad i = 1 \ldots N$$

where $A(x)$, $B(x)$, and $\frac{A(x)}{\sqrt{B(x)}}$ are convex functions.

Then there exists a $\hat{t}$ for which the optimal solution of problem (1) is the same as the optimal portfolio of problem (2):

$$\max A(x) - \hat{t}B(x) \qquad (2)$$
$$g_i(x) \le 0, \qquad i = 1 \ldots N$$

Proof:

$\nabla \frac{A(x)}{\sqrt{B(x)}} |_x = \frac{1}{\sqrt{B(x)}} \left(A'(x) - 0.5B'(x)\frac{A(x)}{B(x)}\right)$; let x^* be optimal in (1), $\mathrm{a} = \frac{1}{\sqrt{B(x^*)}}$,

$\mathrm{b} = 0.5\mathrm{a}\frac{A(x^*)}{B(x^*)}$, π^1 be the optimal dual vector corresponding to x^*.

Then the part of the KKT system involving objective for (1) is

$$\mathrm{a}A'(x^*) - \mathrm{b}B'(x^*) + \sum_{\mathrm{i}} \pi^1{}_i g_i'(x) = 0.$$

By taking $\pi^2 = \frac{1}{a}\pi^1$, $\hat{t} = \frac{b}{a}$, we get the same part of the KKT system for (2).

Scaling π^1 by a positive number keeps complementary slackness and sign restrictions on duals satisfied.

QED.

Theorem 2:
Let S be a convex set defined by a finite number of inequalities, and let $h(\lambda)$ be defined as follows: $h(\lambda) = \arg\max_{h\in S} r^{\mathrm{T}}h - \lambda h^T V h.$

Then the function $SR(\lambda) = \dfrac{r^{\mathrm{T}}h(\lambda)}{\sqrt{h^T(\lambda)Vh(\lambda)}}$ is unimodal, i.e., monotonically non-decreasing to the left of the optimal region, and monotonically non-increasing to the right of it.

Proof:

From Theorem 1 it follows that there exists $\lambda^* = \arg\max_{\lambda} SR(\lambda)$. Let $\lambda_1 \neq \lambda^*$, λ_2 lie strictly between λ^* and λ_1.

Then there exists $0 \le \gamma \le 1$ for which $r^{\mathrm{T}}h(\lambda_2) = \gamma r^{\mathrm{T}}h(\lambda_1) + (1-\gamma)r^{\mathrm{T}}h(\lambda^*)$.

Define $h_C = \gamma h(\lambda_1) + (1-\gamma)h(\lambda^*)$. Notice that

$r^{\mathrm{T}}h(\lambda_2) = r^{\mathrm{T}}\ h_C$, $h_C \in S$ by convexity of S, and hence $\sqrt{h^T(\lambda_2)Vh(\lambda_2)} \le \sqrt{h_C^T V h_C}$, by the argmax property of $h(\lambda_2)$.

By concavity, $\sqrt{h_C^T V h_C} \le \gamma\sqrt{h^T(\lambda_1)Vh(\lambda_1)} + (1-\gamma)\sqrt{h^T(\lambda^*)Vh(\lambda^*)}$.

Therefore:

$$SR(\lambda_2) = \frac{r^{\mathrm{T}}h(\lambda_2)}{\sqrt{h^T(\lambda_2)Vh(\lambda_2)}} \ge \frac{r^{\mathrm{T}}h_C}{\sqrt{h_C^T V h_C}} \ge \frac{\gamma r^{\mathrm{T}}h(\lambda_1) + (1-\gamma)r^{\mathrm{T}}h(\lambda^*)}{\gamma\sqrt{h^T(\lambda_1)Vh(\lambda_1)} + (1-\gamma)\sqrt{h^T(\lambda^*)Vh(\lambda^*)}},$$

Since $\lambda^* = \arg\max_{\lambda} SR(\lambda)$,

$$\frac{r^{\mathrm{T}}h(\lambda^*)}{\sqrt{h^T(\lambda^*)Vh(\lambda^*)}} \ge \frac{r^{\mathrm{T}}h(\lambda_1)}{\sqrt{h^T(\lambda_1)Vh(\lambda_1)}},$$ and hence

$$\frac{\gamma r^{\mathrm{T}}h(\lambda_1) + (1-\gamma)r^{\mathrm{T}}h(\lambda^*)}{\gamma\sqrt{h^T(\lambda_1)Vh(\lambda_1)} + (1-\gamma)\sqrt{h^T(\lambda^*)Vh(\lambda^*)}} \ge \frac{r^{\mathrm{T}}h(\lambda_1)}{\sqrt{h^T(\lambda_1)Vh(\lambda_1)}} = SR(\lambda_1)$$

and $SR(\lambda^*) \ge SR(\lambda_2) \ge SR(\lambda_1)$.

QED.

Theorem 3: Suppose we need to solve the following problem:

$$\begin{aligned} &\text{Maximize} \quad \frac{\mu' h}{\sqrt{h' V h}} \\ &s.t. \qquad Ah \geq b \qquad\qquad (3) \\ &e' h = 1 \end{aligned}$$

We can get the optimal solution of (3) by the following procedure.

First, solve the following linear programming problem:

$$\begin{aligned} &\text{Maximize} \quad \mu' h \\ &s.t. \qquad Ah \geq b \qquad\qquad (3') \\ &e' h = 1. \end{aligned}$$

If the optimum solution in (3') has a positive objective function value, then solve convex quadratic programming problem (4):

$$\begin{aligned} &\text{Minimize} \quad g' V g \\ &s.t. \qquad \mu' g = 1, \\ &Ag - \zeta b \geq 0, \\ &e' g - \zeta = 0, \qquad \zeta \geq 0 \qquad\qquad (4) \end{aligned}$$

<u>Proof</u> (equivalence of (3) and (4)):

A: Let *h* be optimal in (3), and $\mu' h$ =1/ζ >0; $h' V h$ =a>0. Then, its objective value is $f_1 = 1/(\zeta \sqrt{a})$. Let *g*=ζ h. Then (*g*, ζ) is feasible in (4), and its objective value is $f_2 = \zeta^2 a = 1/f_1^2$.

B: Let (*g*, ζ) be optimal in (4), with objective f_2 =a>0.

Then, *h* = g/ζ is feasible in (3) and its objective value is $f_1 = 1/\sqrt{f_2}$.

QED.

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